INDEX-DEPENDENT PARAMETERS OF LAGUERRE AND RELATED POLYNOMIAL SETS[1]

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ABSTRACT. It is known that linearity of the function $\gamma(n)$ is sufficient for the set $\{L_n^{(\gamma(n))}(x)\}$ of generalized Laguerre polynomials to be of type zero as defined by I. M. Sheffer. We prove here that linearity is also necessary. This result is exhibited as a special case in the broader context of generalized Appell representations introduced by R. P. Boas, Jr. and R. C. Buck.

1. Statement of result. Let $\{p_n(x)\}$ be a simple polynomial set possessing a generalized Appell representation as defined by Boas and Buck [1, p. 18]. That is, it is generated by a relation of the form

$$G(t)\Phi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)t^n \tag{1}$$

where

$$\Phi(t) = \sum_{n=0}^{\infty} \phi_n t^n \qquad (\phi_n \neq 0), \tag{2}$$

$$G(t) = \sum_{n=0}^{\infty} g_n t^n \qquad (g_0 \neq 0), \tag{3}$$

and

$$H(t) = \sum_{n=1}^{\infty} h_n t^n \qquad (h_1 \neq 0). \tag{4}$$

We say that $\{p_n(x)\} \in [\Phi]$ where $[\Phi]$ denotes the subclass of all such sets whose generalized Appell representations involve a fixed $\Phi(t)$. In the above expansions we are not concerned with questions of convergence, and our entire discussion is in the language of formal power series.

Suppose further that

$$(1-t)^{-\alpha}G(t)\Phi(xH(t)) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)t^n \tag{5}$$

Received by the editors October 6, 1969.

AMS Subject Classifications. Primary 3340; Secondary 3320.

Key Words and Phrases. Generalized Appell representation, Laguerre polynomials, zero type sets.

[1] This research was partially supported (first author) by National Science Foundation Grant GP-9140.

generates the set $\{p_n^{(\alpha)}(x)\}$ which evidently reduces to $\{p_n(x)\}$ when $\alpha=0$. Then let $\{p_n^{(\gamma(n))}(x)\}$ be the still more general set obtained by replacing the parameter α of $\{p_n^{(\alpha)}(x)\}$ by an arbitrary function $\gamma(n)$ of the index n. Observe that (5) is a generalized Appell representation and we are thus assured [1, pp. 18–19] that $\{p_n^{(\alpha)}(x)\}$, as well as $\{p_n^{(\gamma(n))}(x)\}$, is a simple polynomial set.

The first author [2, equation (4)] has shown that $\{p_n^{(\gamma(n))}(x)\}\in[\Phi]$ when $\gamma(n)=\alpha+\beta n$, α and β being arbitrary constants. That is, in order that $\{p_n^{(\gamma(n))}(x)\}\in[\Phi]$ it is sufficient that the function $\gamma(n)$ be linear for $n\geqq 1$. The value of $\gamma(0)$ is irrelevant since $p_0^{(\alpha)}(x)=g_0\phi_0$, as is seen by setting $t=0$ in (5), and so this polynomial is actually independent of the parameter α. The object of this paper is to prove the necessity of linearity and thereby extend the above result to the following

THEOREM. *In order that* $\{p_n^{(\gamma(n))}(x)\}\in[\Phi]$ *it is necessary and sufficient that the function* $\gamma(n)$ *be linear for* $n\geqq 1$.

An application to Laguerre polynomials will be discussed in §3 below. Indeed, it was consideration of that special case which suggested the theorem.

2. **Proof of theorem.** We start the proof of necessity by relating certain of the coefficients of the polynomials

$$p_n(x) = c_{n,n}x^n + c_{n,n-1}x^{n-1} + \cdots \tag{6}$$

to those of

$$p_n^{(\gamma(n))}(x) = \bar{c}_{n,n}x^n + \bar{c}_{n,n-1}x^{n-1} + \cdots . \tag{7}$$

Specifically, it is immediate from (1) and (5) that

$$(1-t)^{-\alpha}G(t)\Phi(xH(t)) = \left\{\sum_{n=0}^{\infty}\binom{-\alpha}{n}(-1)^n t^n\right\}\left\{\sum_{n=0}^{\infty} p_n(x)t^n\right\},$$

or

$$p_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{-\alpha}{k}(-1)^k p_{n-k}(x).$$

Replacing α by $\gamma(n)$ in the above, we have then

$$p_n^{(\gamma(n))}(x) = p_n(x) + \gamma(n)p_{n-1}(x) + \cdots$$

which, in view of (6), becomes

$$p_n^{(\gamma(n))}(x) = c_{n,n}x^n + (c_{n,n-1} + \gamma(n)c_{n-1,n-1})x^{n-1} + \cdots.$$

Comparison of (7) with this reveals that for $n \geqq 0$

$$\tilde{c}_{n,n} = c_{n,n} \tag{8}$$

and for $n \geqq 1$

$$\tilde{c}_{n,n-1} = c_{n,n-1} + \gamma(n)c_{n-1,n-1}. \tag{9}$$

Next, define for $n \geqq 1$ the two sequences

$$s_n = \frac{\phi_n}{\phi_{n-1}} \cdot \frac{c_{n,n-1}}{c_{n,n}} \tag{10}$$

and

$$\tilde{s}_n = \frac{\phi_n}{\phi_{n-1}} \cdot \frac{\tilde{c}_{n,n-1}}{\tilde{c}_{n,n}} \tag{11}$$

where the ϕ's are the coefficients in (2). Using (8) and (9) to substitute for the ratio $\tilde{c}_{n,n-1}/\tilde{c}_{n,n}$ in (11) and subtracting (10) from the resulting equation, we find that

$$\tilde{s}_n - s_n = \frac{\phi_n}{\phi_{n-1}} \cdot \frac{\gamma(n)c_{n-1,n-1}}{c_{n,n}}. \tag{12}$$

Now, quoting in what amounts to only slightly different notation a theorem due to the second author [**4**, Theorem 2], we know that $\{p_n(x)\} \in [\Phi]$ implies the identity

$$s_n = s_1 + (s_2 - s_1)(n - 1) \tag{13}$$

for all $n \geqq 1$. Moreover, keeping in mind that we are proving the necessity part of our theorem, we have that $\{p_n^{(\gamma(n))}(x)\} \in [\Phi]$ and so

$$\tilde{s}_n = \tilde{s}_1 + (\tilde{s}_2 - \tilde{s}_1)(n - 1) \tag{14}$$

when $n \geqq 1$. Subtract (13) from (14) and replace the resulting terms $\tilde{s}_n - s_n$, $\tilde{s}_1 - s_1$, and $\tilde{s}_2 - s_2$ by the equivalent expressions indicated in (12). This yields

$$\begin{aligned} &\frac{\phi_n}{\phi_{n-1}} \cdot \frac{\gamma(n)c_{n-1,n-1}}{c_{n,n}} \\ &\qquad = \frac{\phi_1}{\phi_0} \cdot \frac{\gamma(1)c_{0,0}}{c_{1,1}} + \left(\frac{\phi_2}{\phi_1} \cdot \frac{\gamma(2)c_{1,1}}{c_{2,2}} - \frac{\phi_1}{\phi_0} \cdot \frac{\gamma(1)c_{0,0}}{c_{1,1}}\right)(n - 1) \end{aligned} \tag{15}$$

for $n \geqq 1$.

Finally, in the proof of the theorem just quoted it has been pointed out that $c_{n,n}=\phi_n g_0 h_1^n$ where g_0 and h_1 are the initial coefficients in (3) and (4). Substituting then for the c's in (15) we find that it reduces to

$$\gamma(n) = 2\gamma(1) - \gamma(2) + (\gamma(2) - \gamma(1))n.$$

That is, for all $n \geqq 1$, $\gamma(n)=\alpha+\beta n$ where α and β are constants, and the proof is complete.

3. **A special case.** If $\Phi(t)=\exp(t)$ in (1), the corresponding generalized Appell subclass [exp] to which $\{p_n(x)\}$ belongs consists of the so-called zero type sets introduced by Sheffer [6]. The set $\{L_n(x)\}$ of simple Laguerre polynomials is, for example, of type zero. In fact, it is the special case when $\alpha=0$ of the set $\{L_n^{(\alpha)}(x)\}$ of generalized Laguerre polynomials generated by [**5**, p. 242]

$$(1 - t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n.$$

Recently Carlitz [3] showed that, in addition, the set $\{L_n^{(\alpha+\beta n)}(x)\}$ is of type zero for arbitrary α and β. The question naturally arises as to whether there are nonlinear functions $\gamma(n)$ of the index n such that $\{L_n^{(\gamma(n))}(x)\}$ is of type zero. With our theorem, we are now able to answer this question in the negative. That is, the set $\{L_n^{(\gamma(n))}(x)\}$ of generalized Laguerre polynomials is of type zero if and only if $\gamma(n)$ is linear for $n \geqq 1$.

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