AN EXAMPLE IN THE WEIL THEORY OF
MEASURABLE GROUPS

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ABSTRACT. According to a theorem of Andre Weil, if a group G possesses an invariant measure which satisfies certain conditions, in particular measurability of the map $(x, y) \to (x, xy)$ of $G \times G$, then G has a locally bounded Hausdorff topology making G a topological group. We offer a simple counterexample to show the need for the above stated condition.

Introduction. A *measurable group* is a σ-finite measure space (G, M, m) such that (a) m is not identically zero, (b) G is a group, (c) the σ-ring M and the measure m are both left invariant, (d) the map S of $G \times G$ given by $S(x, y) = (x, xy)$ is measurability preserving [1, p. 257]. The group is called *separated* if (e) for each $x \in G$ there exists $A \in M$ such that $m(xA \triangle A) > 0$, where $\triangle$ is symmetric difference. Weil's theorem states that for a separated measurable group there exists a locally bounded Hausdorff topology on G making it a topological group, with $m(E) > 0$ for E open and $m(E) < \infty$ for E bounded [1, p. 273–274].

In this note we give a simple example which illustrates the need for the peculiar measurability hypothesis (d). It will be a system (G, M, m) such that (a), (b), (c), and (e) hold, while (d) fails. We show that there exists at least one set of the form $\{x : m(xA \triangle A) < \epsilon\}$, with $A \in M$, which is not in the original σ-ring M, contradicting Theorem C on p. 272 of [1].

I am indebted to Professor W. W. Comfort for suggesting in 1966 that I compute this example (in a somewhat different connection).

1. The group. Let G be the set of pairs (a, t), where a is real and $t = \pm 1$, with group operation $(a, t)(b, s) = (a + tb, ts)$. This is isomorphic to the group generated by translations of R, plus inversion $Ix = -x$. It also has a matrix representation:

$$(a, t) \sim \begin{vmatrix} 1 & a \\ 0 & t \end{vmatrix}.$$

Note that if

$$(b, s) \sim \begin{vmatrix} 1 & b \\ 0 & s \end{vmatrix},$$

then

$$\begin{vmatrix} 1 & b \\ 0 & s \end{vmatrix} \times \begin{vmatrix} 1 & a \\ 0 & t \end{vmatrix} = \begin{vmatrix} 1 & a + tb \\ 0 & st \end{vmatrix}$$

so the product matrix corresponds to $(a, t)(b, s)$ as defined above. This makes it easy to see that G is a group with identity $(0, 1)$ and inverse $(a, t)^{-1} = (-ta, t)$.

The σ-ring M on G is defined as follows: if A is a Borel set in R, $A_0 = A \times \{1\} \cup A \times \{-1\}$. It is easy to check that M is a σ-ring. We define the measure m by $m(A_0) = l(A)$, where l is Lebesgue measure. M is invariant under left translation, for an easy calculation yields $(b, t)A_0 = (b+tA)_0$. To show m is left invariant, $m((b, t)A_0) = m(b+tA)_0 = l(b+tA) = l(tA) = l(A) = m(A_0)$ (recall that $t = \pm 1$).

It is obvious that if a Weil topology were to exist, it could not be Hausdorff. For in that topology every set in the neighborhood base is measurable [1, Theorem C, p. 272], while it follows from the definition of M that for any real a, $(a, 1)$ and $(a, -1)$ belong to exactly the same measurable sets.

2. **Main results.** It is obvious that our system satisfies axioms (a), (b) and (c). That (e) also holds while (d) fails requires proof.

PROPOSITION 1. *If* $(b, t) \neq (0, 1)$, *then there exists* $A_0 \in M$ *such that* $m((b, t)A_0 \triangle A_0) > 0$.

PROOF. First, suppose $t = 1$. Let $A = [0, 1]$. An easy calculation yields $(b, 1)[0, 1]_0 = (b + [0, 1])_0 = [b, 1+b]_0$, and so $m((b, t)A_0 \triangle A_0) = l([b, 1+b] \triangle [0, 1]) > 0$. If $t = -1$ and $b \neq 1$, we again let $A = [0, 1]$, getting $(b, -1)[0, 1]_0 = (b - [0, 1])_0 = [-1+b, b]_0$, and

$$m((b, t)A_0 \triangle A_0) = l([-1 + b, b] \triangle [0, 1]) > 0.$$

If $t = -1$ and $b = 1$, let $A = [-1, 1]$. Then

$$(1, -1)[-1, 1]_0 = (1 - [-1, 1])_0 = [0, 2]_0$$

and

$$m((b, t)A_0 \triangle A_0) = l([0, 2] \triangle [-1, 1]) > 0.$$

PROPOSITION 2. *Axiom* (d) *for measurable groups fails.*

FIRST PROOF. Suppose axiom (d) held. Then it follows from Proposition 1 and preceding remarks that G is a separated measurable group, and hence by Weil's Theorem there exists a Hausdorff Weil

topology for G. But this contradicts the remark made at the end of §1.

SECOND PROOF. Since this is intended to be a *simple* example, we should not base our main result on the relatively deep theorem of Weil. In this proof we show first that sets in the product measure space $M \times M$ have a certain symmetry property, and then show that this symmetry is not preserved under the map S of axiom (d).

First, if A and B are Borel sets in R, then

$$A_0 \times B_0 = (A \times \{1\} \cup A \times \{-1\}) \times (B \times \{1\} \cup B \times \{-1\})$$
$$= (A \times \{1\} \times B \times \{1\}) \cup (A \times \{1\} \times B \times \{-1\})$$
$$\cup (A \times \{-1\} \times B \times \{1\}) \cup (A \times \{-1\} \times B \times \{-1\}).$$

Now if $t = \pm 1$ and $s = \pm 1$, let

$$E(t, s) = \{(x, y) \in R \times R : (x, t, y, s) \in A_0 \times B_0\}.$$

Then $E(1, 1) = E(1, -1) = E(-1, 1) = E(-1, -1)$. It is easy, if messy, to check that this symmetry is preserved under arbitrary unions and intersections, hence pertains to all sets in the product σ-ring $M \times M$. Now if A and B are Borel sets in R, an easy calculation gives

$$S(A_0 \times B_0) = \{(a, -1, a - b, 1) : a \in A, b \in B\}$$
$$\cup \{(a, 1, a + b, -1) : a \in A, b \in B\}$$
$$\cup \{(a, -1, a - b, 1) : a \in A, b \in B\}$$
$$\cup \{(a, 1, a + b, 1) : a \in A, b \in B\}.$$

If $A = B = [0, 1]$, then

$$E(1, 1) = \{(x, x + y) : x \in [0, 1], y \in [0, 1]\},$$
$$E(-1, 1) = \{(x, x - y) : x \in [0, 1], y \in [0, 1]\},$$

with $(0, -1) \in E(-1, 1)$ and $(0, -1) \notin E(1, 1)$.

PROPOSITION 3. *There exists $A_0 \in M$ and $\epsilon > 0$ such that*

$$\{(a, t) : m((a, t)A_0 \triangle A_0) < \epsilon\} \notin M.$$

PROOF. Let $A = [0, 1]$, $\epsilon = 1/3$, and E be the set in question. Then

$$E = \{(a, 1) : m((a, 1)[0, 1]_0 \triangle [0, 1]_0) < 1/3\}$$
$$\cup \{(a, -1) : m((a, -1)[0, 1]_0 \triangle [0, 1]_0) < 1/3\}$$
$$= \{(a, 1) : l([a, 1 + a] \triangle [0, 1]) < 1/3\}$$
$$\cup \{(a, -1) : l([-1 + a, a] \triangle [0, 1]) < 1/3\}$$
$$= \{(a, 1) : a \in (-1/3, 1/3)\} \cup \{(a, -1) : a \in (2/3, 4/3)\}.$$

Since M consists exclusively of "symmetric" sets of the form $A \times \{1\} \cup A \times \{-1\}$, it follows that $E \notin M$.

REFERENCES

1. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR **11**, 504.

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