FREDHOLM TOEPLITZ OPERATORS[1]

R. G. DOUGLAS AND DONALD SARASON

ABSTRACT. Whether a Toeplitz operator on H^2 is Fredholm is shown to depend only on the local behavior of the inducing function.

Let L^2 and L^∞ denote the Lebesgue spaces of square integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let H^2 and H^∞ denote the corresponding Hardy spaces. For ϕ in L^∞, the *Toeplitz operator induced by* ϕ is the operator T_ϕ on H^2 defined by $T_\phi f = P(\phi f)$; here, P stands for the orthogonal projection in L^2 with range H^2. A great deal of effort has been devoted to finding criteria, in terms of the behavior of ϕ, for the invertibility of T_ϕ; a summary of most of the known results can be found in [2]. All such criteria involve the global behavior of ϕ. In contrast to this, we show here that whether or not T_ϕ is a Fredholm operator depends, in a sense, only on the local behavior of ϕ.

We recall that a Hilbert space operator is said to be *left-Fredholm* if it is left-invertible modulo the compact operators, or, equivalently, if it has a closed range and a finite dimensional null space. An operator is said to be *right-Fredholm* if its adjoint is left-Fredholm, *semi-Fredholm* if it is either right-Fredholm or left-Fredholm, and *Fredholm* if it is both right-Fredholm and left-Fredholm.

THEOREM. *Let ϕ be a function in L^∞. Assume that for each point λ on the unit circle there is an open subarc A of the unit circle containing λ and a function f in L^∞ such that $f|A = \phi|A$ and T_f is left-Fredholm. Then T_ϕ is left-Fredholm.*

This theorem was suggested by, and contains as a corollary, a recent result of H. Widom and the first author [3].

The proof uses the following three known results, which are due, respectively, to Hartman and Wintner, Coburn, and Rabindranathan.

LEMMA A [5]. *If f is in L^∞ and f is not essentially bounded away from zero, then T_f is not bounded below.*

Received by the editors December 31, 1969.

AMS 1970 *subject classifications*. Primary 47B35; Secondary 46J15.

Key words and phrases. Toeplitz operator, Fredholm operator, Hardy spaces.

[1] Research of both authors supported in part by the National Science Foundation. Both authors are fellows of the Alfred P. Sloan Foundation.

LEMMA B [1]. *If f is in L^∞, then either T_f or T_f^* has a trivial kernel.*

LEMMA C [8, LEMMA 1]. *If f is in L^∞ and $|f|=1$ a.e., then T_f is left-invertible if and only if* $\operatorname{dist}(f, H^\infty)<1$.

We first show, using a standard argument, that it is enough to treat the case where $|\phi|=1$ a.e. By Lemma B, if T_f is semi-Fredholm then it is either left-invertible or right-invertible. In either case it follows from Lemma A that f is essentially bounded away from 0. The hypotheses of the theorem thus imply that ϕ is essentially bounded away from 0. Therefore there is an invertible function h in H^∞ such that $|h|=|\phi|$ a.e. [7, pp. 53–54]. Let $\psi=h^{-1}\phi$. Then $|\psi|=1$ a.e. and $T_\phi=T_\psi T_h$. The operator T_h is invertible, and hence T_ϕ is left-Fredholm if and only if T_ψ is left-Fredholm. Moreover, it is evident that ψ satisfies the basic hypothesis of the theorem. It will therefore suffice to prove the theorem under the additional assumption that $|\phi|=1$ a.e.

Let C denote the space of continuous complex valued functions on the unit circle. We recall that $H^\infty+C$, the linear span of H^∞ and C, is a closed subalgebra of L^∞ [6, Theorem 2].

LEMMA 1. *Let ϕ be a function in L^∞ such that $|\phi|=1$ a.e. Then T_ϕ is left-Fredholm if and only if* $\operatorname{dist}(\phi, H^\infty+C)<1$.

To prove this, let χ denote the identity function on the unit circle ($\chi(z)\equiv z$). The inequality $\operatorname{dist}(\phi, H^\infty+C)<1$ is equivalent to the existence of a nonnegative integer n such that $\operatorname{dist}(\chi^n\phi, H^\infty)<1$, and hence, by Lemma C, to the existence of a nonnegative integer n such that $T_{\chi^n\phi}$ is left-invertible. If the latter happens then, because T_{χ^n} is Fredholm and $T_{\chi^n\phi}=T_\phi T_{\chi^n}$, the operator T_ϕ must be left-Fredholm. On the other hand, if T_ϕ is left-Fredholm but not left-invertible, then by Lemma B it is Fredholm of index $n>0$. In this case, since T_{χ^n} has index $-n$, the operator $T_{\chi^n\phi}$ has index 0 and so is invertible by Lemma B. The proof of Lemma 1 is complete.

Let X be the maximal ideal space of L^∞. For f in L^∞, the Gelfand transform of f will be denoted by $\hat{f}$. The Gelfand transforms of the subalgebras H^∞ and $H^\infty+C$ will be denoted by $(H^\infty)\hat{}$ and $(H^\infty+C)\hat{}$.

For λ on the unit circle, let X_λ denote the fiber of X above λ, that is, the set of maximal ideals in X whose corresponding multiplicative linear functionals assign to χ the value λ. We note that, since the Gelfand transforms of the functions in C are constant on each X_λ, the restriction algebras $(H^\infty)\hat{}\,|X_\lambda$ and $(H^\infty+C)\hat{}\,|X_\lambda$ are identical.

Kenneth Hoffman has pointed out to the authors the following unexpected property of $H^\infty+C$: *If f is in L^∞ and $\hat{f}|X_\lambda$ is in* $(H^\infty)\hat{}\,|X_\lambda$

for each λ, *then* f *is in* $H^\infty+C$. This follows from a well-known theorem of Bishop on the decomposition of function algebras into antisymmetric algebras [4], together with the observation that every set of antisymmetry of $(H^\infty+C)\hat{}$ is contained in a single fiber. We need a mild refinement of Hoffman's result which follows from the following addendum Glicksberg has provided to Bishop's theorem.

LEMMA D [4, p. 419]. *Let* Y *be a compact Hausdorff space and* B *a closed subalgebra of* $C(Y)$ *containing the constants. Let* $\mathcal{K}$ *be the family of maximal sets of antisymmetry of* B. *Then for any* g *in* $C(Y)$,

$$\operatorname{dist}(g, B) = \max_{K\in\mathcal{K}} \operatorname{dist}(g \mid K, B \mid K).$$

Lemma D has the following immediate consequence.

LEMMA 2. *If* ϕ *is in* L^∞, *then*

$$\operatorname{dist}(\phi, H^\infty + C) = \max_{|\lambda|=1} \operatorname{dist}(\hat{\phi} \mid X_\lambda, (H^\infty)\hat{} \mid X_\lambda).$$

We can now complete the proof of the theorem in a few lines. Let ϕ satisfy the hypotheses of the theorem and the additional condition that $|\phi|=1$ a.e. By Lemmas 1 and 2 it will suffice to show that $\operatorname{dist}(\hat{\phi}|X_\lambda, (H^\infty)\hat{}\,|X_\lambda)<1$ for each λ on the unit circle. Fix λ, and choose an open subarc A containing λ and a function f in L^∞ such that $f|A=\phi|A$ and T_f is left-Fredholm. As observed earlier, the function f must be essentially bounded away from 0, so there is an invertible function h in H^∞ such that $|h|=|f|^{-1}$ a.e. Since T_h is invertible and $T_{hf}=T_fT_h$, the operator T_{hf} is left-Fredholm. Therefore, by Lemma 1, $\operatorname{dist}(hf, H^\infty+C)<1$. Hence $\operatorname{dist}(\hat{h}\hat{f}|X_\lambda, (H^\infty)\hat{}\,|X_\lambda)<1$. But $\hat{f}|X_\lambda=\hat{\phi}|X_\lambda$, and thus $\operatorname{dist}(\hat{h}\hat{\phi}|X_\lambda, (H^\infty)\hat{}\,|X_\lambda)<1$. Finally, since $|h|=1$ a.e. on A, we have $|\hat{h}|=1$ on X_λ, so because h^{-1} is in H^∞ it follows that $\operatorname{dist}(\hat{\phi}|X_\lambda, (H^\infty)\hat{}\,|X_\lambda)<1$. The theorem is proved.

The above argument clearly establishes a slightly stronger result than was stated: *Let* ϕ *be in* L^∞. *Assume that for each* λ *on the unit circle there is an* f *in* L^∞ *such that* $\hat{f}|X_\lambda=\hat{\phi}|X_\lambda$ *and* T_f *is left-Fredholm. Then* T_ϕ *is left-Fredholm.* As an immediate corollary we conclude that if there exists for each λ on the unit circle a function f in L^∞ such that $\hat{f}|X_\lambda=\hat{\phi}|X_\lambda$ and T_f is Fredholm, then T_ϕ is Fredholm.

REFERENCES

1. L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288. MR **34** #1846.

2. R. G. Douglas, *On the spectrum of Toeplitz and Wiener-Hopf operators*, Proc. Conference Abstract Spaces and Approximation (Oberwohlfach, 1968), I. S. N. M., vol. 10, Birkhäuser Verlag, Basel, 1969.

3. R. G. Douglas and H. Widom, *Toeplitz operators with locally sectorial symbols*, J. Math. Mech. (to appear).

4. I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc. **105** (1962), 415–435. MR **30** #4164.

5. P. Hartman and A. Wintner, *On the spectra of Toeplitz's matrices*, Amer. J. Math. **72** (1950), 359–366. MR **12**, 187.

6. H. Helson and D. Sarason, *Past and future*, Math. Scand. **21** (1967), 5–16. MR **38** #5282.

7. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962. MR **24** #A2844.

8. M. Lee and D. Sarason, *The spectra of some Toeplitz operators*, J. Math Anal. Appl. (to appear).

State University of New York at Stony Brook, Stony Brook, New York 11790

University of California, Berkeley, California 94720