

FREDHOLM TOEPLITZ OPERATORS¹

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ABSTRACT. Whether a Toeplitz operator on H^2 is Fredholm is shown to depend only on the local behavior of the inducing function.

Let L^2 and L^∞ denote the Lebesgue spaces of square integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let H^2 and H^∞ denote the corresponding Hardy spaces. For ϕ in L^∞ , the *Toeplitz operator induced by ϕ* is the operator T_ϕ on H^2 defined by $T_\phi f = P(\phi f)$; here, P stands for the orthogonal projection in L^2 with range H^2 . A great deal of effort has been devoted to finding criteria, in terms of the behavior of ϕ , for the invertibility of T_ϕ ; a summary of most of the known results can be found in [2]. All such criteria involve the global behavior of ϕ . In contrast to this, we show here that whether or not T_ϕ is a Fredholm operator depends, in a sense, only on the local behavior of ϕ .

We recall that a Hilbert space operator is said to be *left-Fredholm* if it is left-invertible modulo the compact operators, or, equivalently, if it has a closed range and a finite dimensional null space. An operator is said to be *right-Fredholm* if its adjoint is left-Fredholm, *semi-Fredholm* if it is either right-Fredholm or left-Fredholm, and *Fredholm* if it is both right-Fredholm and left-Fredholm.

THEOREM. *Let ϕ be a function in L^∞ . Assume that for each point λ on the unit circle there is an open subarc A of the unit circle containing λ and a function f in L^∞ such that $f|_A = \phi|_A$ and T_f is left-Fredholm. Then T_ϕ is left-Fredholm.*

This theorem was suggested by, and contains as a corollary, a recent result of H. Widom and the first author [3].

The proof uses the following three known results, which are due, respectively, to Hartman and Wintner, Coburn, and Rabindranathan.

LEMMA A [5]. *If f is in L^∞ and f is not essentially bounded away from zero, then T_f is not bounded below.*

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LEMMA B [1]. *If f is in L^∞ , then either T_f or T_f^* has a trivial kernel.*

LEMMA C [8, LEMMA 1]. *If f is in L^∞ and $|f| = 1$ a.e., then T_f is left-invertible if and only if $\text{dist}(f, H^\infty) < 1$.*

We first show, using a standard argument, that it is enough to treat the case where $|\phi| = 1$ a.e. By Lemma B, if T_f is semi-Fredholm then it is either left-invertible or right-invertible. In either case it follows from Lemma A that f is essentially bounded away from 0. The hypotheses of the theorem thus imply that ϕ is essentially bounded away from 0. Therefore there is an invertible function h in H^∞ such that $|h| = |\phi|$ a.e. [7, pp. 53-54]. Let $\psi = h^{-1}\phi$. Then $|\psi| = 1$ a.e. and $T_\phi = T_\psi T_h$. The operator T_h is invertible, and hence T_ϕ is left-Fredholm if and only if T_ψ is left-Fredholm. Moreover, it is evident that ψ satisfies the basic hypothesis of the theorem. It will therefore suffice to prove the theorem under the additional assumption that $|\phi| = 1$ a.e.

Let C denote the space of continuous complex valued functions on the unit circle. We recall that $H^\infty + C$, the linear span of H^∞ and C , is a closed subalgebra of L^∞ [6, Theorem 2].

LEMMA 1. *Let ϕ be a function in L^∞ such that $|\phi| = 1$ a.e. Then T_ϕ is left-Fredholm if and only if $\text{dist}(\phi, H^\infty + C) < 1$.*

To prove this, let χ denote the identity function on the unit circle ($\chi(z) \equiv z$). The inequality $\text{dist}(\phi, H^\infty + C) < 1$ is equivalent to the existence of a nonnegative integer n such that $\text{dist}(\chi^n \phi, H^\infty) < 1$, and hence, by Lemma C, to the existence of a nonnegative integer n such that $T_{\chi^n \phi}$ is left-invertible. If the latter happens then, because T_{χ^n} is Fredholm and $T_{\chi^n \phi} = T_\phi T_{\chi^n}$, the operator T_ϕ must be left-Fredholm. On the other hand, if T_ϕ is left-Fredholm but not left-invertible, then by Lemma B it is Fredholm of index $n > 0$. In this case, since T_{χ^n} has index $-n$, the operator $T_{\chi^n \phi}$ has index 0 and so is invertible by Lemma B. The proof of Lemma 1 is complete.

Let X be the maximal ideal space of L^∞ . For f in L^∞ , the Gelfand transform of f will be denoted by \hat{f} . The Gelfand transforms of the subalgebras H^∞ and $H^\infty + C$ will be denoted by $(H^\infty)^\wedge$ and $(H^\infty + C)^\wedge$.

For λ on the unit circle, let X_λ denote the fiber of X above λ , that is, the set of maximal ideals in X whose corresponding multiplicative linear functionals assign to χ the value λ . We note that, since the Gelfand transforms of the functions in C are constant on each X_λ , the restriction algebras $(H^\infty)^\wedge|_{X_\lambda}$ and $(H^\infty + C)^\wedge|_{X_\lambda}$ are identical.

Kenneth Hoffman has pointed out to the authors the following unexpected property of $H^\infty + C$: *If f is in L^∞ and $\hat{f}|_{X_\lambda}$ is in $(H^\infty)^\wedge|_{X_\lambda}$*

for each λ , then f is in $H^\infty + C$. This follows from a well-known theorem of Bishop on the decomposition of function algebras into antisymmetric algebras [4], together with the observation that every set of antisymmetry of $(H^\infty + C)^\wedge$ is contained in a single fiber. We need a mild refinement of Hoffman's result which follows from the following addendum Glicksberg has provided to Bishop's theorem.

LEMMA D [4, p. 419]. *Let Y be a compact Hausdorff space and B a closed subalgebra of $C(Y)$ containing the constants. Let \mathcal{K} be the family of maximal sets of antisymmetry of B . Then for any g in $C(Y)$,*

$$\text{dist}(g, B) = \max_{K \in \mathcal{K}} \text{dist}(g | K, B | K).$$

Lemma D has the following immediate consequence.

LEMMA 2. *If ϕ is in L^∞ , then*

$$\text{dist}(\phi, H^\infty + C) = \max_{|\lambda|=1} \text{dist}(\hat{\phi} | X_\lambda, (H^\infty)^\wedge | X_\lambda).$$

We can now complete the proof of the theorem in a few lines. Let ϕ satisfy the hypotheses of the theorem and the additional condition that $|\phi| = 1$ a.e. By Lemmas 1 and 2 it will suffice to show that $\text{dist}(\hat{\phi} | X_\lambda, (H^\infty)^\wedge | X_\lambda) < 1$ for each λ on the unit circle. Fix λ , and choose an open subarc A containing λ and a function f in L^∞ such that $f|_A = \phi|_A$ and T_f is left-Fredholm. As observed earlier, the function f must be essentially bounded away from 0, so there is an invertible function h in H^∞ such that $|h| = |f|^{-1}$ a.e. Since T_h is invertible and $T_{hf} = T_f T_h$, the operator T_{hf} is left-Fredholm. Therefore, by Lemma 1, $\text{dist}(hf, H^\infty + C) < 1$. Hence $\text{dist}(\hat{h}\hat{f} | X_\lambda, (H^\infty)^\wedge | X_\lambda) < 1$. But $\hat{f}|_{X_\lambda} = \hat{\phi}|_{X_\lambda}$, and thus $\text{dist}(\hat{h}\hat{\phi} | X_\lambda, (H^\infty)^\wedge | X_\lambda) < 1$. Finally, since $|h| = 1$ a.e. on A , we have $|\hat{h}| = 1$ on X_λ , so because h^{-1} is in H^∞ it follows that $\text{dist}(\hat{\phi} | X_\lambda, (H^\infty)^\wedge | X_\lambda) < 1$. The theorem is proved.

The above argument clearly establishes a slightly stronger result than was stated: *Let ϕ be in L^∞ . Assume that for each λ on the unit circle there is an f in L^∞ such that $\hat{f}|_{X_\lambda} = \hat{\phi}|_{X_\lambda}$ and T_f is left-Fredholm. Then T_ϕ is left-Fredholm.* As an immediate corollary we conclude that if there exists for each λ on the unit circle a function f in L^∞ such that $\hat{f}|_{X_\lambda} = \hat{\phi}|_{X_\lambda}$ and T_f is Fredholm, then T_ϕ is Fredholm.

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