A CHARACTERIZATION OF SEMILOCAL INERTIAL COEFFICIENT RINGS

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ABSTRACT. A commutative ring R with identity is called an *inertial coefficient ring* if every finitely generated R-algebra A with A/N separable over R contains a separable subalgebra S such that $S+N=A$, where N is the Jacobson radical of A. Thus inertial coefficient rings are those commutative rings R for which a generalization of the Wedderburn Principal Theorem holds for suitable R-algebras. Our purpose is to prove that a commutative ring with only finitely many maximal ideals is an inertial coefficient ring (if and) only if it is a finite direct sum of Hensel rings.

Azumaya has shown that any Hensel ring is an inertial coefficient ring [**2**, Theorem 33]. It follows easily [**5**, Proposition 3.2] that any finite direct sum of Hensel rings is an inertial coefficient ring. That an inertial coefficient ring with only finitely many maximal ideals is a direct sum of Hensel rings has been proved by Ingraham under the additional hypotheses that R is noetherian and either all residue class fields of R are perfect or R is an integrally closed domain [**5**, Corollary 3.12]. Brown subsequently relaxed this final condition to R being a domain whose integral closure R' in the quotient field of R has the property that $R'/\mathrm{rad}(R')$ is separable over R [**3**, Theorem 3]. Our present methods simplify these earlier proofs while establishing the more general result.

All rings will contain an identity and all subrings will contain the identity of the overring. Throughout R will denote a commutative *semilocal* ring; that is, R is a commutative ring ***without chain conditions*** containing only a finite number of maximal ideals. We will call R *connected* if it has no idempotents but 0 and 1. Clearly every semilocal ring is a finite direct sum of connected rings. By an R-algebra we mean a ring along with a ring homomorphism of R into the center of the ring. An R-algebra is called *separable* if it is projective as a module over its enveloping algebra [**1**, p. 369]. From now on, we will let A denote a commutative R-algebra, finitely generated and faithful as an R-module, with Jacobson radical N. A separable subalgebra S of

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A such that $S+N=A$ is called an *inertial subalgebra* of A. The basic properties of inertial subalgebras and inertial coefficient rings can be found in [**5**].

We begin by deriving a numerical criterion for the existence of inertial subalgebras which will be used to show that any semilocal connected inertial coefficient ring is local, that is, contains a unique maximal ideal. For any maximal ideal $\mathfrak{m}$ of R; A possesses only a finite number of maximal ideals M lying over $\mathfrak{m}$, i.e. with $M \cap R = \mathfrak{m}$. For any such M, A/M can be considered a field extension of $R/\mathfrak{m}$ of finite dimension $[A/M:R/\mathfrak{m}]$. Let $\mu_{\mathfrak{m}}(A) = \sum_M [A/M:R/\mathfrak{m}]$ where M runs over those maximal ideals of A contracting to the fixed maximal ideal $\mathfrak{m}$ of R.

LEMMA 1. *Let A be a finitely generated, commutative, projective, faithful R-algebra, where R is connected. If A contains an inertial subalgebra S, then $\mu_{\mathfrak{m}}(A) = \mu_{\mathfrak{m}'}(A)$ for any two maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ in R.*

PROOF. For any maximal ideal M of A, $S+M=A$ since $S+N=A$, so $S/(S \cap M) \simeq A/M$. Also distinct maximal ideals M and M' of A contract to distinct maximal ideals of S, since $S \cap M = S \cap M'$ would give

$$\begin{aligned} S/(S \cap M) &= S/(S \cap M \cap M') \simeq [S + (M \cap M')]/(M \cap M') \\ &= A/(M \cap M') \simeq A/M \oplus A/M', \end{aligned}$$

a contradiction. These remarks, along with the fact that every maximal ideal of S is the contraction of a maximal ideal of A (A is integral over S), imply that $\mu_{\mathfrak{m}}(S) = \mu_{\mathfrak{m}}(A)$. But $S/\mathfrak{m}S$ is separable over the field $R/\mathfrak{m}$ since S is separable over R, implying that $S/\mathfrak{m}S$ is semisimple and so is isomorphic to $\oplus \sum_M S/(S \cap M)$ where M runs over the finite collection of maximal ideals of A lying over $\mathfrak{m}$. Hence $\mu_{\mathfrak{m}}(A) = \mu_{\mathfrak{m}}(S) = [S/\mathfrak{m}S:R/\mathfrak{m}]$. Now S is a projective R-module [**5**, Proposition 2.8], so for every maximal ideal $\mathfrak{m}$ of R, $S_{\mathfrak{m}}$ has finite rank as a free $R_{\mathfrak{m}}$-module. Moreover this rank is seen to equal $[S/\mathfrak{m}S:R/\mathfrak{m}] = \mu_{\mathfrak{m}}(A)$. But because R is connected, the rank of S localized at any maximal ideal does not depend on the choice of maximal ideal [**4**, Chapter 2, §4], so $\mu_{\mathfrak{m}}(A) = \mu_{\mathfrak{m}'}(A)$, proving the lemma.

Notice that the proof of Lemma 1 holds for any ground ring R, regardless of how many maximal ideals it possesses.

PROPOSITION 2. *Every connected, semilocal inertial coefficient ring R is local.*

PROOF. The proof is patterned closely after that of Theorem 3.6 of

[5], so we omit some details. The role taken by Corollary 2.11 in that proof is here played by our Lemma 1.

Under the assumption that R is connected and contains more than one maximal ideal, we construct a finitely generated, free, commutative R-algebra A with A/N R-separable but such that $\mu_{\mathfrak{m}}(A) \neq \mu_{\mathfrak{m}'}(A)$ for some maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ of R. Thus, by Lemma 1, A cannot contain an inertial subalgebra, so R is not an inertial coefficient ring.

We treat two cases.

Case 1. $2 \notin \operatorname{rad}(R)$. Let $\mathfrak{m}$ be a maximal ideal of R with $2 \notin \mathfrak{m}$. Assuming R has other maximal ideals, one can choose an element α lying in all the maximal ideals of R except $\mathfrak{m}$. Set $A = R[x]/(x^2 - \alpha)$. Clearly A is a finitely generated, free, commutative R-algebra. Furthermore, for any maximal ideal M of A, either $A/M \simeq R/(R \cap M)$ (if $\alpha \in M$) or A/M is an at most two-dimensional field extension of $R/\mathfrak{m}$ (if $\alpha \notin M$). Since the characteristic of $R/\mathfrak{m}$ is not 2, we conclude that each A/M and therefore A/N is separable over R. However, one easily sees by examining $A/\mathfrak{m}A$ and $A/\mathfrak{m}'A$ that $\mu_{\mathfrak{m}}(A) = 2$ while $\mu_{\mathfrak{m}'}(A) = 1$ for any maximal ideal $\mathfrak{m}' \neq \mathfrak{m}$. Thus R is not an inertial coefficient ring by Lemma 1.

Case 2. $2 \in \operatorname{rad}(R)$. It follows from Lemma 3.10 of [5] (the assumption of noetherian therein being unnecessary) that we can assume R contains a primitive cube root of 1. Then, supposing R contains more than one maximal ideal $\mathfrak{m}$, we again select α to be an element lying in every maximal ideal of R except $\mathfrak{m}$. This time setting $A = R[x]/(x^3 - \alpha)$, we see that for any maximal ideal M of A, A/M is a field extension of dimension one or three over $R/(R \cap M)$. (The existence of a primitive cube root of 1 is used here. See p. 88 of [5].) Since the characteristic of $R/(R \cap M)$ is two, it follows that A/N is R-separable. One checks that $\mu_{\mathfrak{m}'}(A) = 1$ for any maximal ideal $\mathfrak{m}' \neq \mathfrak{m}$, while $A/\mathfrak{m}A$ being separable over $R/\mathfrak{m}$ implies $\mu_{\mathfrak{m}}(A) = 3$. Hence R is not an inertial coefficient ring and the proof is complete.

Next we treat the local case. Let R be a local ring with unique maximal ideal $\mathfrak{m}$. For $f \in R[x]$, denote by $\bar{f}$ the element of $(R/\mathfrak{m})[x]$ obtained from f by reducing its coefficient modulo $\mathfrak{m}$. R is called a *Hensel ring* [2, p. 137] if, for every monic polynomial $f \in R[x]$ such that $\bar{f} = g_0 h_0$ in $(R/\mathfrak{m})[x]$, where g_0 and h_0 are monic and relatively prime, there exist monic polynomials g and h in $R[x]$ with $f = gh$ and $\bar{g} = g_0$, $\bar{h} = h_0$. For our purposes, the important fact is that a local ring R is Hensel if for every finitely generated, commutative, and faithful R-algebra A and any ideal I of A with $\mathfrak{m}A \subseteq I \subseteq N$, idempotents can be lifted from A/I to A [2, Theorem 19].

PROPOSITION 3. *Let R be a local ring. The following statements are equivalent*:

(a) *R is an inertial coefficient ring*;

(b) *Every finitely generated, commutative, faithful R-algebra A with A/N separable over R is a finite direct sum of local rings*;

(c) *R is a Hensel ring.*

PROOF. That (c) implies (a) is essentially the content of Theorem 33 of [2].

(a) implies (b). Let A be a finitely generated, commutative, faithful R-algebra with A/N separable. By the proof of Proposition 3.3 of [5], A is itself an inertial coefficient ring since R is. Now A can be written as a finite direct sum of connected rings, each of which is an inertial coefficient ring [5, Corollary 3.4]. Hence each connected component of A is local by Proposition 2, proving (b).

(b) implies (c). It suffices to show that for any finitely generated, commutative, faithful R-algebra A, we can lift any idempotent from A/I to A, where I is any ideal with $\mathfrak{m}A \subseteq I \subseteq N$. We proceed by letting c be any element of A whose image $\bar{c}$ in A/I is an idempotent. Consider the subalgebra $R[c]$ of A generated by c. It is easy to verify that the natural homomorphism from A onto A/I when restricted to $R[c]$ gives rise to the exact sequence

$$0 \to R[c] \cap I \to R[c] \to (R/\mathfrak{m})[\bar{c}] \to 0.$$

Now $(R/\mathfrak{m})[\bar{c}]$ is a homomorphic image of $(R/\mathfrak{m})[x]/(x^2-x) \simeq (R/\mathfrak{m}) \oplus (R/\mathfrak{m})$ and so is R-separable. Since $(R/\mathfrak{m})[\bar{c}]$ is semisimple, $R[c] \cap I \supseteq \operatorname{rad}(R[c])$ while the opposite inclusion follows from the integrality of A over $R[c]$. Therefore $R[c]$ is a finitely generated, commutative, faithful R-algebra such that $R[c]/\operatorname{rad}(R[c])$ is separable. Applying (b) not to A but to $R[c]$, we see that $R[c]$ is a direct sum of either one or two local rings according as $R[c]/\operatorname{rad}(R[c]) \simeq (R/\mathfrak{m})[\bar{c}]$ is $R/\mathfrak{m}$ or $(R/\mathfrak{m}) \oplus (R/\mathfrak{m})$. In either case it is clear that there must exist an idempotent in $R[c] \subseteq A$ mapping onto $\bar{c}$. This proves Proposition 3.

As an immediate consequence of Propositions 2 and 3 we have

THEOREM. *A semilocal ring (not necessarily noetherian) is an inertial coefficient ring (if and) only if it is a finite direct sum of Hensel rings.*

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