# ON A CONJECTURE OF E. GRANIRER CONCERNING THE RANGE OF AN INVARIANT MEAN

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Abstract. The purpose of this paper is to prove the following conjecture of E. Granirer: if $S$ is an infinite right cancellation left amenable semigroup then for each left invariant mean $\phi$ of $S$, $\{\phi(A): A \subset S\} = [0, 1]$.

Let $S$ be a semigroup with discrete topology, $m(S)$ the space of bounded real functions on $S$ with the sup norm. $\phi \in m(S)^*$ is a mean if $\|\phi\| = 1$, and $(\phi, f) \geqq 0$ whenever $f \geqq 0$. A mean $\phi$ is said to be left invariant if $(\phi, l_s f) = (\phi, f)$ for $s \in S$ and $f \in m(S)$, where $l_s f \in m(S)$ is defined by $(l_s f)(s_1) = f(ss_1)$. If $m(S)$ has a left invariant mean then we say $S$ is left amenable.

Each mean $\phi$ on $m(S)$ can be considered as a finite additive measure on the family of all subsets of $S$. For $A \subset S$, $\phi(\chi_A)$ will also be denoted by $\phi(A)$. Clearly, if $\phi$ is a mean then the range of $\phi$, $\{\phi(A): A \subset S\}$, is a subset of $[0, 1]$. The purpose of this paper is to prove the following.

Theorem. *Let $S$ be an infinite right cancellation left amenable semigroup. Then the range of each left invariant mean on $m(S)$ is the whole $[0, 1]$ interval.*

Granirer stated this theorem as a conjecture in [**3**]. There he was able to prove it (in a stronger form) for all cases except when $S$ is a so-called "AB group". An infinite torsion group $S$ is an AB group if (a) $S$ is amenable, (b) each infinite subgroup of $S$ is not locally finite, cf. [**3**].

Each mean $\phi$ on $m(S)$ corresponds to a unique probability measure $\mu_\phi$ on $\beta S$, the Stone-Čech compactification of the discrete set $S$. The correspondence is characterized by $(\phi, f) = \int_{\beta S} f^- d\mu_\phi$, where $f \in m(S)$ and $f^-$ denotes its continuous extension to $\beta S$. If $B \subset S$, $B^-$ will denote the closure of $B$ in $\beta S$. Sets of the form $B^-$, $B \subset S$, are closed-open in $\beta S$ and they form a topological open basis for $\beta S$.

For each $s \in S$, we have a continuous mapping $s^\sim$ of $S$ into $\beta S$ defined by $s^\sim(s_1) = ss_1$, $s_1 \in S$. $s^\sim$ has a unique continuous extension to $\beta S$. The extended mapping will also be denoted by $s^\sim$. If $S$ is actually a group then, for each $s \in S$, $s^\sim$ is a homeomorphism from $\beta S$ onto $\beta S$ (cf. [**1**, Lemma 2.1]). Moreover, $(s_1^{-1}s_2)^\sim = (s_1^\sim)^{-1}s_2^\sim$ and $e^\sim$ = the iden-

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tity function from $\beta S$ onto $\beta S$, where $s_1, s_2 \in S$ and $e$ is the identity of $S$.

LEMMA 1. *Let $S$ be an infinite group, $w \in \beta S$, and $s_1, s_2 \in S$, $s_1 \neq s_2$. Then $\tilde{s_1} w \neq \tilde{s_2} w$.*

PROOF. By the remark above, we may assume that $s_1 = e$ and $s_2 = s \neq e$. By Lemmas 1 and 2 of [2], there exist a positive integer $k$ and subsets $A_1, A_2, \cdots, A_k$ of $S$ such that (1) $\bigcup_{i=1}^k A_i = S$, (2) $A_i \cap A_j = \emptyset$ if $i \neq j$, and (3) $sA_i \subset A_{i+1}$ if $i \leqq k-1$ and $sA_k \subset A_1$. Note that by (1) $\beta S = \bigcup_{i=1}^k A_i^-$. Thus, if $w \in \beta S$ then $w \in A_i^-$ for some $i$, say, $i = 1$. Then, by (3), $\tilde{s} w \in A_2^-$. Since (2) implies that $A_1^- \cap A_2^- = \emptyset$, we conclude $\tilde{s} w \neq w$.

LEMMA 2. *Let $S$ be an infinite amenable group. If $\phi$ is a left invariant mean on $m(S)$ then $\mu_\phi(\{w\}) = 0$ for each $w \in \beta S$.*

PROOF. Let $w \in \beta S$. Choose a subset $\{s_1, \cdots, s_n\}$ of $S$, where $s_i \neq s_j$, if $i \neq j$. Then, by Lemma 1, $\tilde{s_i} w \neq \tilde{s_j} w$ if $i \neq j$. It is clear that we can choose a closed-open neighborhood $A^-$ of $w$ such that $\tilde{s_i} A^- \cap \tilde{s_j} A^- = \emptyset$ if $i \neq j$. Denote the characteristic function of $A$ in $S$ by $\chi_A$. Then

$$\begin{aligned} 1 &= \mu_\phi(\beta S) \geqq \sum_{i=1}^n \mu_\phi(\tilde{s_i} A^-) \\ &= \sum_{i=1}^n \mu_\phi((s_i A)^-) \qquad \text{(cf. [1, Lemma 2.1])} \\ &= \sum_{i=1}^n \phi(s_i A) = \sum_{i=1}^n \phi(l_{s_i^{-1}} \chi_A) \\ &= n\phi(A) = n\mu_\phi(A^-) \\ &\geqq n\mu_\phi(\{w\}). \end{aligned}$$

Since $n$ can be arbitrarily big, $\mu_\phi(\{w\}) = 0$.

LEMMA 3. *Let $X$ be an infinite discrete set and $\phi$ be a mean on $m(X)$ such that $\mu_\phi(\{w\}) = 0$ for each $w \in \beta X$. Then $\{\phi(A) : A \subset X\} = [0, 1]$.*

PROOF. Note first that $\mu_\phi$ is nonatomic. Indeed, if $\mu_\phi$ has atoms then there is a compact atom $K$. We may cover $K$ by a finite number of open sets $U_1, \cdots, U_n$ with $\mu_\phi(U_i) < \mu_\phi(K)$ for $i = 1, 2, \cdots, n$. Since $K$ is an atom, $\mu_\phi(U_i \cap K) = 0$, and hence, $\mu_\phi(K) = 0$, a contradiction. Consequently, $\mu_\phi$ is nonatomic and by Liapounoff's convexity theorem, cf. [4],[1] $\{\mu_\phi(\Omega) : \Omega$ runs over Borel subsets of $\beta X\} = [0, 1]$.

[1] Many thanks are due to R. Bourgin for referring this paper to us.

Let $\Omega$, a Borel subset of $\beta X$, and $\epsilon>0$ be given. Since $\mu_\phi$ is regular, there exist a closed set $\Gamma$ and an open set $\Lambda$ of $\beta X$ such that $\Lambda\supset\Omega\supset\Gamma$ and $u_\phi(\Lambda\backslash\Gamma)<\epsilon$. Since $\Gamma$ is compact, we can find a closed-open subset $A^-\subset\beta X$ such that $\Lambda\supset A^-\supset\Gamma$, and hence, $|\mu_\phi(A^-)-\mu_\phi(\Omega)|<\epsilon$. Thus we conclude that $\{\phi(A):A\subset X\}$ is dense in $[0, 1]$.

Let $\lambda\in(0, 1)$ be given. Choose a sequence $(\lambda_n)$ in $(0, 1)$ such that $\lambda_{2n-1}>\lambda_{2n+1}>\lambda>\lambda_{2n+2}>\lambda_{2n}$, $n=1, 2, \cdots$, and $\lim_n \lambda_{2n-1}=\lambda=\lim_n \lambda_{2n}$. Since $\{\phi(A):A\subset X\}$ is dense in $[0, 1]$, we can choose a set $A_1\subset X$ such that $\lambda_3<\phi(A_1)<\lambda_1$. Similarly, since $\{\phi(A):A\subset A_1\}$ is dense in $[0, \phi(A_1)]$, there exists $A_2\subset A_1$ such that $\lambda_2<\phi(A_2)<\lambda_4$. Again, since $\{\phi(A):A\subset A_1\backslash A_2\}$ is dense in $[0, \phi(A_1\backslash A_2)]$, we can choose $B_3\subset A_1\backslash A_2$ such that $\lambda_5<\phi(A_2\cup B_3)<\lambda_3$. Set $A_3=A_2\cup B_3$. Continue this process. We get a sequence of subsets $(A_n)$ of $X$ such that

$$A_{2n-1}\supset A_{2n+1}\supset A_{2n+2}\supset A_{2n}$$

and

$$\lambda_{2n+1}<\phi(A_{2n-1})<\lambda_{2n-1}, \qquad \lambda_{2n}<\phi(A_{2n})<\lambda_{2n+2},$$

for $n=1, 2, \cdots$. Let $A=\bigcap_{n=1}^\infty A_{2n-1}$. Then $A_{2n-1}\supset A\supset A_{2n}$, and hence, $\lambda_{2n-1}>\phi(A)>\lambda_{2n}$, $n=1, 2, \cdots$. Thus $\phi(A)=\lim_n \lambda_n=\lambda$ and the proof is completed.

Proof of the Theorem. By Theorem 1 and Lemma 4 of [**3**], we may assume that $S$ is an infinite amenable group. The Theorem is then a direct consequence of Lemma 2 and Lemma 3.

## References

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