WEAK A-CONVEX ALGEBRAS

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ABSTRACT. Necessary and sufficient conditions are given in terms of E' that a weak topology $w(E, E')$ on an algebra E be A-convex. The main condition is that each element g of E' contain a weakly closed subspace L of finite codimension such that g is bounded on all multiplicative translates of L. For weak topologies, A-convexity (which assumes only separate continuity of multiplication) is equivalent to joint continuity of multiplication.

Let E be an algebra, E' a total subspace of the dual of E and $w(E, E')$ the weak topology of E determined by E'. The purpose of this paper is to determine necessary and sufficient conditions that $w(E, E')$ be A-convex. Warner [**4**] has given a necessary and sufficient condition that $w(E, E')$ be locally m-convex and also a necessary and sufficient condition that multiplication be jointly $(w(E, E'))$ continuous. One of the equivalent forms of our condition is that $w(E, E')$ is A-convex (which requires only the separate continuity of multiplication) if and only if multiplication is jointly $(w(E, E'))$ continuous. Thus, all weak topological algebras (joint continuity of multiplication) are already A-convex. A-convex algebras, which include the locally m-convex algebras, were introduced in [**2**]. In §2 the basic properties are given along with some examples. The main results are given in §3.

2. ***A*-convex algebras.** Throughout this note, E will denote an algebra, E' a total subspace of the dual of E and $w(E, E')$ the weak topology on E induced by E'. The proofs of the results given here may be found in [**2**].

(2.1) DEFINITION. A subset V of E is called A-convex if V is absolutely convex, absorbing and for each $x \in E$, V absorbs xV and Vx.

The inverse image of an A-convex set under a homomorphism is A-convex, as is the image of an A-convex set under a surjective homomorphism.

(2.2) DEFINITION. An A-convex algebra is an algebra E together with a topology on E whose neighborhood system at zero has a basis of A-convex sets.

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(2.3) Definition. A seminorm p on E is called m-absorbing if for all x in E there are constants M_x and N_x with

(i) $p(xy) \leqq M_x p(y)$, for all y in E; and

(ii) $p(yx) \leqq N_x p(y)$, for all y in E.

Thus, an A-convex algebra is an algebra E with a topology determined by a family of m-absorbing seminorms. A-convexity is preserved with respect to taking subspaces, products and quotients. Each A-convex algebra can be topologically and algebraically embedded in an A-convex algebra with identity. It is clear from the definition that multiplication is separately continuous.

(2.4) Example. Any locally m-convex (hence also Banach) algebra is A-convex.

(2.5) Example. Let $C[0, 1]$ denote the algebra of continuous real-valued functions on $[0, 1]$ (with pointwise operations). A norm on $C[0, 1]$ is given by

$$p(f) = \sup\{ |f(x)\phi(x)| : x \in [0, 1]\},$$

where $\phi(x) = x$, $0 \leqq x \leqq \frac{1}{2}$ and $\phi(x) = 1 - x$, $\frac{1}{2} < x \leqq 1$. $(C[0, 1], p)$ is a normed linear space which is A-convex (not locally m-convex). An A-convex algebra which is normable is called an A-normed algebra. The space $(C[0, 1], p)$ is not complete.

(2.6) Example. Let $C_b(\boldsymbol{R})$ denote the algebra of bounded continuous complex-valued functions on $\boldsymbol{R}$ (pointwise operations). Let $C_0^+(\boldsymbol{R})$ denote the set of strictly positive real-valued continuous functions on $\boldsymbol{R}$ which vanish at infinity. For each $\phi \in C_0^+(\boldsymbol{R})$, let

$$p_\phi(f) = \sup\{ |f(x)\phi(x)| : x \in \boldsymbol{R}\}, \qquad f \in C_b(\boldsymbol{R}).$$

Then p_ϕ is a seminorm and the topology β determined by $\{p_\phi : \phi \in C_0^+(\boldsymbol{R})\}$ is an A-convex topology on $C_b(\boldsymbol{R})$. This so-called weighted space is a complete A-convex algebra with identity which is not locally m-convex (see [2], [6]).

(2.7) Theorem. *An algebra E with a locally convex linear topology for which multiplication is separately continuous is A-convex if and only if it is isomorphic to a subalgebra of a product of A-normed algebras.*

The relationship between A-convex and locally m-convex algebras is given by the fact that a barrelled A-convex algebra is locally m-convex.

For the remainder of this paper the following notations will be used: For a linear functional g on E, $K(g)$ will denote the kernel of g. The polar of a set V in E', taken in E, will be denoted by V°.

3. **Weak A-convex algebras.** In the proof of the main two theorems the following results will be needed. The proof of Lemma 3.1 may be found in [**4**] and the proof of Lemma 3.2 is omitted.

(3.1) LEMMA. *Let V be a $w(E, E')$-neighborhood of zero. Then $L = \bigcap \{K(v): v \in V^0\}$ is a $w(E, E')$-closed subspace of finite codimension.*

(3.2) LEMMA. *Let g be a linear functional on E and L a subspace of $K(g)$. Then $EL \subseteq K(g)$ is equivalent to the property*

(*) *for all x in E there is a constant M_x such that $|g(xL)| \leqq M_x$. Also, $LE \subseteq K(g)$ is equivalent to*

(**) *for all x in E there is a constant N_x such that $|g(Lx)| \leqq N_x$.*

If g is a linear functional on E and L is a subspace such that (*) and (**) hold, then we say that g is bounded on the (multiplicative) translates of L.

(3.3) THEOREM. *Let E be an algebra and E' a total subspace of the dual of E. Then $w(E, E')$ is A-convex if and only if for all g in E', $K(g)$ contains a weakly closed subspace L of finite codimension such that g is bounded on all translates of L.*

PROOF. Let $w(E, E')$ be A-convex and g be in E'. Then $\{g\}^0$ contains an A-convex weakly closed neighborhood V of zero. Let $L = \bigcap \{K(v): v \in V^0\}$. By Lemma 3.1, L is a weakly closed subspace of finite codimension. Clearly g is in V^0, $L \subseteq V^{00}$ and $V = V^{00}$ so $L \subseteq K(g)$.

For x in E, the A-convexity of V insures the existence of constants M_x and N_x such that $|g(xL)| \leqq M_x$ and $|g(Lx)| \leqq N_x$. Hence g is bounded on all translates of L and the condition is necessary.

Let g $(\neq 0)$ be in E' and L a weakly closed subspace of finite codimension with $L \subseteq K(g)$ and g bounded on all translates of L. It suffices to show that $\{g\}^0$ contains an A-convex weak neighborhood of zero. By the induced map theorem there exists a unique continuous linear functional $\bar{g}$ on $F = E/L$ such that $g = \bar{g} \circ \phi_1$, where ϕ_1 denotes the quotient map from E to F. Since F is a finite dimensional Hausdorff space, it is normable. We may assume that the norm on F is chosen so that $\bar{g}(\hat{V}) = \{z: |z| \leqq 1\}$ where $\hat{V}$ is the unit ball in F. The map ϕ_1 is a continuous linear functional so $V = \phi_1^{-1}(\hat{V})$ is an absolutely convex absorbing neighborhood of zero. Then

$$|g(V)| = |\bar{g}(\phi_1(V))| = |\bar{g}(\hat{V})| \leqq 1, \quad \text{so } V \subseteq \{g\}^0.$$

We now show that V is A-convex: For x in E, $xL \subseteq K(g)$ by Lemma

3.2 and $[xL]^-$ is a closed subspace contained in $K(g)$. By the induced map theorem there is a unique continuous linear functional g^* on $H=E/[xL]^-$ with $g=g^* \circ \phi_2$ where ϕ_2 is the quotient map of E to H. The map t_x of F to H defined by $t_x(\hat{y})=[xy]^-$ is a well-defined linear map from the finite dimensional space F to H. Thus, t_x is continuous and the image of F is a finite dimensional Hausdorff space. The image of F under t_x is normable and the norm may be chosen so that $|g^*(V^*)| \leqq 1$, where V^* denotes the unit ball in H. It follows from a standard theorem that there exists a constant M_x such that

$$\|t_x(\hat{y})\|_2 \leqq M_x\|\hat{y}\|, \quad \text{for all } \hat{y} \text{ in } F,$$

where $\| \quad \|_2$ denotes the norm on $t_x(F)$ and $\| \quad \|$ denotes the norm on F. If y is in V, $\hat{y}$ is in $\hat{V}$ and $\|[xy]^-\|_2 \leqq M_x$. Thus,

$$|g^*(M_x^{-1}[xy]^-)| \leqq 1 \quad \text{so} \quad |g(M_x^{-1}xy)| \leqq 1.$$

This shows that $xV \subseteq M_xV$. Similarly, there is a constant N_x with $Vx \subseteq N_xV$. Hence V is A-convex and the condition is sufficient.

The result of Theorem 3.3 gives another interesting consequence: A-convexity is equivalent to joint $(w(E, E'))$-continuity of multiplication. Hence any weak topological algebra (joint continuity) is an A-convex algebra.

(3.4) THEOREM. *Let E be an algebra. Then $w(E, E')$ is A-convex if and only if the multiplication of E is jointly $(w(E, E'))$ continuous.*

PROOF. Theorem 3.3 combined with Lemma 3.2 give the result that $w(E, E')$ is A-convex if and only if for all g in E', $K(g)$ contains a weakly closed subspace L of finite codimension with $EL \subseteq K(g)$ and $LE \subseteq K(g)$. From a theorem of Warner [**4**, Theorem 2] this is equivalent to the joint continuity of multiplication.

(3.5) COROLLARY. *Let E be a topological algebra with respect to a weak topology. Then E is A-convex.*

The following problem remains unsolved:

(3.6) PROBLEM. Is there an example of an algebra E and a subspace E' of its dual such that $w(E, E')$ is A-convex but not locally m-convex?

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