POSITIVELY CURVED DEFORMATIONS OF INVARIANT RIEMANNIAN METRICS

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ABSTRACT. Let K_γ denote the sectional curvature function of the Riemannian metric γ on a manifold M. Suppose M admits no metric γ invariant under the action of a compact group G and having $K_\gamma > 0$. It is shown that a G-invariant metric $\gamma(0)$ with $K_{\gamma(0)} \geqq 0$ cannot be embedded in a 1-parameter family $\gamma(t)$ for which $[dK_{\gamma(t)}/dt]_{t=0}$ is positive wherever $K_{\gamma(0)}$ is zero.

The sectional curvature of the Riemannian metric γ on the manifold M is a function K_γ on the bundle $P(M)$ of tangent 2-planes to M. Denote by $Z_\gamma \subseteq P(M)$ the set of zeros of K_γ. A smooth 1-parameter family $\gamma(t)$ will be called *positive* if $K_{\gamma(0)}$ is nonnegative and if $[dK_{\gamma(t)}/dt]_{t=0}$ is positive on $Z_{\gamma(0)}$. If M is compact, $K_{\gamma(\epsilon)}$ is positive for small positive ϵ. Berger [2] showed that if M is compact, a family on M for which $\gamma(0)$ is a product metric cannot be positive. The following theorem, combined with the results cited after its proof, gives other examples of metrics which cannot be embedded in positive families.

THEOREM. *If M is a compact manifold which admits no metric of positive sectional curvature invariant under a given compact group G of diffeomorphisms, then a family $\gamma(t)$ of metrics on M for which $\gamma(0)$ is G-invariant cannot be positive.*

PROOF. We prove the contrapositive. Suppose that $\gamma(t)$ is G-invariant, and define

$$\delta(t) = \int_G g^*\gamma(t)dg,$$

where dg is the Haar measure on G normalized so that G has total measure 1. $\delta(0) = \gamma(0)$, and $\delta(t)$ is G-invariant for each t. Denoting by K_* the differential at $\delta(0) = \gamma(0)$ of the mapping $\gamma \mapsto K_\gamma$, we have

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\begin{aligned}
[dK_{\delta(t)}/dt]_{t=0} &= K_*[d\delta(t)/dt]_{t=0} \\
&= K_* \int_G g^*[d\gamma(t)/dt]_{t=0}\, dg \\
&= \int_G K_* g^*[d\gamma(t)/dt]_{t=0}\, dg \\
&= \int_G g^* K_*[d\gamma(t)/dt]_{t=0}\, dg \\
&= \int_G g^*[dK_{\gamma(t)}/dt]_{t=0}\, dg.
\end{aligned}
$$

The second and third lines are equal because K_* is a linear partial differential operator. The g^* in the last two lines refers to the induced action of G on $P(M)$. The relation $K_*g^* = g^*K_*$, valid for any diffeomorphism g, expresses the naturality of the operation $\gamma \mapsto K_\gamma$.

Since $Z_{\gamma(0)}$ is invariant under this action, the integrand in the last integral is positive on $Z_{\gamma(0)}$ for each $g \in G$. It follows that $[dK_{\delta(t)}/dt]_{t=0}$ is positive on $Z_{\gamma(0)}$, and $K_{\delta(\epsilon)}$ is positive for small positive ϵ. Since $\delta(\epsilon)$ is G-invariant, the contrapositive theorem is proven.

The interest of this result rests on the fact that many compact homogeneous spaces G/H are known to admit no G-invariant metric of positive curvature. If the linear isotropy group of G acting on G/H is irreducible, then any G-invariant metric is proportional to the normal one, and Berger [1] has shown that every homogeneous space with normal metric of positive curvature is diffeomorphic to a symmetric space of rank one or to one of two exceptional spaces. Recently, Wallach [3] has shown that if G/H is even-dimensional and admits a G-invariant metric of positive curvature then M is diffeomorphic to a symmetric space of rank one or to one of two other exceptional spaces. It is unknown whether Wallach's exceptional spaces actually admit homogeneous metrics of positive curvature.

References

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