A NOTE ON CONNECTED AND PERIPHERALLY CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper it is proved that under certain conditions on the domain and range spaces an open monotone connected function preserves unicoherentness and hereditary local connectedness. In addition, a monotone-light factorization theorem is proved for certain connected functions and peripherally continuous functions.

1. **Introduction.** Many properties of continuous functions are also possessed by certain noncontinuous functions (see for example [**1**]–[**4**] and [**6**]). This paper is concerned with three of these properties: namely, preservation by an open monotone connected function of unicoherentness and hereditary local connectedness, and factorization of connected and peripherally continuous functions.

Some definitions will now be recalled. Let X and Y be topological spaces. A function $f: X \rightarrow Y$ is connected if whenever A is a connected set in X, then $f(A)$ is connected in Y [**6**, p. 488]. The function f is peripherally continuous if whenever p is a point in X and U and V are open sets containing p and $f(p)$, respectively, there exists an open set W such that $x \in W \subset U$ and f maps the boundary of W into V [**2**, p. 751]. The function f is monotone if for each $y \in Y, f^{-1}(y)$ is connected.

If A is a set, $\mathrm{cl}(A)$ will denote the closure of A and $F(A)$ will denote the boundary of A. The notation $x_n \rightarrow x$ is used for a sequence $\{x_n\}$ converging to x. If A is a collection of sets, A^* will denote the point set union of the sets in A.

2. **Connected functions.** In Theorems 1 and 2 the only assumptions concerning the function f are that f is open in Theorem 1 and f is open and monotone in Theorem 2. These theorems are then used in Theorems 3 and 4 to obtain the results that, under certain conditions, unicoherentness and hereditary local connectedness are preserved by an open monotone connected function.

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THEOREM 1. *If f is an open function from the topological space X onto the topological space Y and $y_n \to y$ in Y, then $f^{-1}(y) \subset \liminf f^{-1}(y_n)$.*

PROOF. If $x \in f^{-1}(y)$ and U is an open set in X such that $x \in U$ and U fails to intersect $f^{-1}(y_n)$ for infinitely many n, then since f is open $f(U)$ is an open set containing y and $f(U)$ fails to contain y_n for infinitely many n. This contradicts $y_n \to y$.

THEOREM 2. *If f is an open monotone function from the topological space X onto the first countable space Y and E is connected in Y, then $f^{-1}(E)$ is connected in X.*

PROOF. Suppose $f^{-1}(E) = H \cup K$, where H and K are separated. Then $E = f(H) \cup f(K)$ and $f(H)$ and $f(K)$ are not separated. Suppose q is a point in $f(H) \cap f(K)$. Then for some r in H and t in K, $f(r) = f(t) = q$. Hence r and t are in $f^{-1}(q)$. Since f is monotone $f^{-1}(q)$ is connected. Therefore either $f^{-1}(q) \subset H$ or $f^{-1}(q) \subset K$ since H and K are separated. But $r \in f^{-1}(q) \subset H$ and $t \in f^{-1}(q) \subset K$. This contradiction shows that $f(H) \cap f(K) = \emptyset$. Thus, one of $f(H)$ and $f(K)$ must contain a limit point of the other. Suppose q is in $f(H)$ and is a limit point of $f(K)$. Since Y is first countable there is a sequence $\{q_n\}$ in $f(K)$ such that $q_n \to q$. By Theorem 1, $f^{-1}(q) \subset \liminf f^{-1}(q_n)$. Hence, if x is in $f^{-1}(q)$, then x is a limit point of K since $f^{-1}(q_n) \subset K$ for all n. But $x \in H$ since $f^{-1}(q) \subset H$. Thus, H and K are not separated. This contradiction shows that $f^{-1}(E)$ is connected.

THEOREM 3. *If f is an open monotone connected function from the unicoherent T_2 continuum X onto the first countable compact T_2 space Y, then Y is a unicoherent continuum.*

PROOF. Since f is a connected function, then Y is connected and hence a continuum by the virtue of being compact. Suppose $Y = H \cup K$, where H and K are continua. Then $X = f^{-1}(H) \cup f^{-1}(K)$. By Theorem 2, $f^{-1}(H)$ and $f^{-1}(K)$ are connected. By Theorem 2.1 of [**4**], $f^{-1}(H)$ and $f^{-1}(K)$ are closed and hence are continua since X is compact. Since X is unicoherent $f^{-1}(H) \cap f^{-1}(K)$ is a continuum. Now $f^{-1}(H) \cap f^{-1}(K) = f^{-1}(H \cap K)$ and since f is connected, $f(f^{-1}(H \cap K)) = H \cap K$ is connected. Thus, $H \cap K$ is a closed connected subset of the compact space Y and is therefore a continuum. This shows that Y is unicoherent.

THEOREM 4. *If f is an open monotone connected function from X onto Y, where X and Y are separable metric continua and X is hereditarily locally connected, then Y is hereditarily locally connected.*

PROOF. By Theorem 2.1 of [7, p. 89] a continuum is hereditarily locally connected if and only if it has no nondegenerate continuum of convergence. Therefore, suppose K is a nondegenerate continuum of convergence in Y. Let $\{K_n\}$ be a sequence of disjoint continua none intersecting K such that $\lim K_n = K$, and let p and q be distinct points in K. By Theorem 7 of [7, p. 11] some subsequence $\{f^{-1}(K_{ni})\}$ of $\{f^{-1}(K_n)\}$ converges to a limiting set L. Since X is compact, $L \neq \emptyset$. Hence, by Theorem 9.1 of [7, p. 14], L is a continuum. Let $x \in f^{-1}(p)$ and U an open set in X containing x. Since f is an open function, $f(U)$ is open and contains p. Therefore $f(U)$ intersects K_n for all but a finite number of n. Hence, U intersects $f^{-1}(K_n)$ for all but a finite number of n, and consequently intersects $f^{-1}(K_{ni})$ for all but a finite number of i. Thus, $x \in L$ and hence $f^{-1}(p) \subset L$. Similarly, $f^{-1}(q) \subset L$. Since neither of $f^{-1}(p)$ and $f^{-1}(q)$ is empty and they are disjoint, it follows that L is nondegenerate. Therefore X contains a nondegenerate continuum of convergence. This contradicts X being hereditarily locally connected. Thus, Y is hereditarily locally connected.

3. **Factorization.** If f is a function from the space X onto the space Y, let X' denote the collection of all components of sets $f^{-1}(y)$, where y varies over Y. The collection X' will be called the component decomposition of X induced by f. Theorems 5 and 6 give a factorization of f analogous to that given in Theorems 3.6 and 3.7 of [1].

THEOREM 5. *If f is a connected function from the compact, separable metric space X onto the regular T_2 space Y, and the component decomposition X' of X induced by f is upper semicontinuous, then f can be factored into the composite $f = f_2 f_1$, where f_1 from X onto X' is monotone and continuous and f_2 from X' onto Y is light and connected.*

PROOF. Define $f_1(x) = C$ if and only if x is a point in C, where C is a component of some $f^{-1}(y)$, y in Y. By [7, p. 127], f_1 is monotone and continuous. Define $f_2(C) = y$ if and only if C is a component of $f^{-1}(y)$. Then f_2 is light since the elements of $f_2^{-1}(y)$ are the components of $f^{-1}(y)$ and these form a totally disconnected set in X'. For if H is any nondegenerate subcollection of $f_2^{-1}(y)$, then H is connected in X' if and only if H^* is connected in X [5, p. 275]. Now H being nondegenerate implies H^* contains more than one component of $f^{-1}(y)$ and hence is not connected. Thus H is not connected and f_2 is therefore light.

By definition of f_1 and f_2, $f = f_2 f_1$. Therefore it remains to show that f_2 is connected.

To this end let A be a connected subset of X'. Then A^* is connected in X. Since f is a connected function $f(A^*)$ is connected. But $f(A^*)$

$=f_2f_1(A^*)=f_2(A)$. Hence, $f_2(A)$ is connected and f_2 is a connected function.

THEOREM 6. *If f is a peripherally continuous function from the compact separable metric space X onto the regular T_2 space Y and the component decomposition X' of X is upper semicontinuous, then f can be factored into the composite $f=f_2f_1$, where f_1 and f_2 are defined as in Theorem 5, f_1 is monotone and continuous, and f_2 is light and peripherally continuous.*

PROOF. That f_1 is monotone and continuous and f_2 is light follows as in Theorem 5.

Since f is peripherally continuous, the components of $f^{-1}(y)$, y in Y, are closed by Theorem 1 of [**3**, p. 639]. Hence the elements of X' are continua since X is compact.

Let g be an element in X' and $f(g)=y$, and let U and V be open in X' and Y containing g and y, respectively. Then U^* is open in X and $g\subset U^*$. Since f is peripherally continuous, for each x in g there is an open set W_x in X such that $x\in W_x\subset U^*$ and $f(F(W_x))\subset V$. The collection $\{W_x : x\in g\}$ is an open covering of g. Since g is compact some finite subcollection $\{W_1, \cdots, W_n\}$ covers g. Let $W=\bigcup_{i=1}^n W_i$. Then $g\subset W\subset U^*$ and $f(F(W))\subset V$.

Let $H\subset X'$ such that k is in H if and only if $k\subset W$. Then H is open in X' and $H\subset U$. Let $h\in F(H)$ and suppose $h\cap F(W)=\emptyset$. Now $h\subset \mathrm{cl}(W)=W\cup F(W)$ and $W\cap F(W)=\emptyset$. Hence, $h\subset W$ and thus $h\in H$. This contradicts $h\in F(H)$. Therefore, $h\cap F(W)\neq\emptyset$. Let $x\in h\cap F(W)$. Then $f(x)=f(h)=f_2(h)$ and since $x\in F(W)$, $f(x)\in V$. Thus, $f_2(h)\in V$ and $f_2(F(H))\subset V$. This shows that f_2 is peripherally continuous.

REMARK. In [**1**] a proof is given that under rather restrictive conditions on the space X, the component decomposition X' of X induced by a peripherally continuous function is indeed upper semicontinuous. If this could be proven under milder restrictions for connected and peripherally continuous functions, then Theorems 5 and 6 would more closely resemble the monotone-light factorization theorem for continuous functions [**7**, p. 143].

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REFERENCES

1. M. R. Hagan, *Upper semi-continuous decompositions and factorization of certain non-continuous transformations*, Duke Math. J. **32** (1965), 679–687. MR **32** #4668.

2. O. H. Hamilton, *Fixed points for certain noncontinuous transformations*, Proc. Amer. Math. Soc. **8** (1957), 750–756. MR **19**, 301.

3. P. E. Long, *Properties of certain non-continuous transformations*, Duke Math. J. **28** (1961), 639–645. MR **24** #A2945.

4. ———, *Connected mappings*, Duke Math. J. **35** (1968), 677–682. MR **38** #2745.

5. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R.I., 1962. MR **27** #709.

6. W. J. Pervin and Norman Levine, *Connected mappings of Hausdorff spaces*, Proc. Amer. Math. Soc. **9** (1958), 488–496. MR **20** #1970.

7. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R.I., 1963. MR **32** #425.

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