AN EXAMPLE ON EMBEDDING UP TO HOMOTOPY TYPE

ELMER REES[1]

ABSTRACT. A finite complex K is constructed with the following property. $K \vee S^r$ embeds in $\boldsymbol{R}^n$ up to homotopy type but K does not.

T. Ganea (see [4, p. 156]) has asked the following

QUESTION. If K is a finite complex dominated by a subset of $\boldsymbol{R}^n$, is there a subcomplex of $\boldsymbol{R}^n$ of the homotopy type of K?

In this note this is answered negatively. A finite complex K_n of dimension $8n-3$ is constructed (for each $n \geqq 3$) which cannot be embedded up to homotopy type in S^{8n}; however, there is a subcomplex of S^{8n} homotopically equivalent to $K_n \vee S^{4n}$.

Consider the complex Stiefel manifold $W_{2n,2}$ of two frames in $\boldsymbol{C}^{2n}$. $W_{2n,2}$ has a cell decomposition $(S^{4n-3} \vee S^{4n-1}) \cup_{\alpha_n} e^{8n-4}$ where $\alpha_n = i_{4n-3} \circ \beta_n + [i_{4n-3}, i_{4n-1}]$ (i_m denotes the inclusion of S^m). Its suspension $SW_{2n,2}$ is homotopically equivalent to $K_n \vee S^{4n}$ where K_n is $S^{4n-2} \cup_{S\beta_n} e^{8n-3}$ and $S^2\beta_n = -[\iota_{4n-1}, \iota_{4n-1}] \in \pi_{8n-3}S^{4n-1}$. All this is contained in [3].

PROPOSITION 1. *K_n does not embed up to homotopy type in S^{8n} if $n \geqq 3$.*

PROOF. If K_n does embed in S^{8n}, it has complement homotopically equivalent to $S^2 \cup_{\gamma_n} e^{4n+1}$ and $S^{4n-3}\gamma_n = S^2\beta_n = -[\iota_{4n-1}, \iota_{4n-1}]$ by the results of [2]. But by Corollary 1.3 of [1], if $n \geqq 3$ then $[\iota_{4n-1}, \iota_{4n-1}]$ is not a $(4n-3)$-fold suspension.

PROPOSITION 2. *$W_{2n,2}$ embeds in S^{8n-1}.*

PROOF. $W_{2n,2}$ is the sphere bundle of an $\boldsymbol{R}^{4n-2}$ bundle ξ over S^{4n-1} and $\xi \oplus \epsilon^2$ is trivial. So $W_{2n,2}$ is included in the total space of ϵ^{4n} which embeds in S^{8n-1}.

COROLLARY. *$K_n \vee S^{4n}$ embeds in S^{8n}.*

PROOF. $W_{2n,2}$ embeds in S^{8n-1} and so its suspension (which is homotopically equivalent to $K_n \vee S^{4n}$, by the above) embeds in S^{8n}.

Received by the editors November 30, 1969.

AMS subject classifications. Primary 5570.

Key words and phrases. Embedding up to homotopy type, complex Stiefel manifolds.

[1] Supported in part by National Science Foundation grant GP-7952X1.

Therefore the complexes K_n have the required properties.

References

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632. MR **25** #2614.

2. G. Cooke, *Embedding certain complexes up to homotopy type in euclidean space*, Ann. of Math. (2) **90** (1969), 144–156.

3. I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres*. I, Proc. London Math. Soc. (3) **4** (1954), 196–218. MR **15**, 892.

4. S. P. Novikov, *The topology summer institute* (Seattle, Wash., 1963), Uspehi Mat. Nauk **20** (1965), no. 1 (121), 147–169 = Russian Math. Surveys **20** (1965), no. 1, 145–167.

Institute for Advanced Study, Princeton, New Jersey 08540.