ON $\mathfrak{g}$-NORMALIZERS AND $\mathfrak{g}$-HYPERCENTER

NOBUO INAGAKI

Abstract. In this note, we shall prove a theorem which is a generalization of the following theorem: Let $G$ be a soluble group, then the intersection of all system normalizers of $G$ is the hypercenter of $G$.

Recently, Carter and Hawkes [1] have generalized the construction of system normalizers of finite soluble groups introducing the concept of the $\mathfrak{g}$-normalizer, and B. Huppert [2] has generalized the hypercenter introducing the concept of the $\mathfrak{g}$-hypercenter. In this note, we shall show the relations between $\mathfrak{g}$-normalizers and the $\mathfrak{g}$-hypercenter, which generalize the theorem [3, VI. 11.11] on the relations between the system normalizers and the hypercenter. The methods of proof of our theorem are similar to those in [3].

1. Definitions. All groups in this note are soluble and finite.

If nonempty formations $\mathfrak{F}(p)$, one for each $p$, are given, the local formation $\mathfrak{F}$ locally defined by $\{\mathfrak{F}(p)\}$ is the class of all groups $G$ such that, whenever $M$ is a chief factor of $G$, of order $p^n$, say, then the automorphisms group induced on $M$ by $G$ belongs to $\mathfrak{F}(p)$. And $M$ is called an $\mathfrak{F}$-central-$p$-chief factor. By Carter and Hawkes [1, p. 177] we can choose $\mathfrak{F}(p)$ such that $\mathfrak{F}(p) \subseteq \mathfrak{g}$. Let $\{S_p\}$ be a set of $p$-complements of $G$, one for each prime $p$ dividing $|G|$, and let $\gamma$ be the Sylow system of $G$ generated by $\{S_p\}$. We write $T_p = S_p \cap C_p$ for each prime $p$ dividing $|G|$, where $C_p$ is the intersection of the centralizers of the $\mathfrak{g}$-central-$p$-chief factors of $G$. The set $T = \{T_p\}$ will be called an $\mathfrak{g}$-system of $G$(see [1]). And then the subgroup $D = \bigcap_p N_\alpha(T_p)$ will be called the $\mathfrak{g}$-normalizer of $G$ (see [1]). Since any two Sylow systems of $G$ are conjugate in $G$, it follows that the $\mathfrak{g}$-normalizers of $G$ form a characteristic conjugacy class of subgroups of $G$. Let $N$ be a normal subgroup of $G$ with a normal chain $e = N_0 \lhd N_1 \lhd \cdots \lhd N_r = N$ where each $N_i$ is normal in $G$. $N$ is called an $\mathfrak{g}$-hypercentral subgroup whenever the following conditions are satisfied

(1) $N_i/N_{i-1}$ is a chief factor of $G$.

(2) If the order of $N_i/N_{i-1}$ is $p^n_i$, then $G/C_\alpha(N_i/N_{i-1}) \subseteq \mathfrak{F}(p_i)$.

As a product of $\mathfrak{g}$-hypercentral normal subgroups is $\mathfrak{g}$-hypercentral, so the product of all $\mathfrak{g}$-hypercentral normal subgroups of $G$
is called the $\mathfrak{g}$-hypercenter of $G$, say $Z_{\mathfrak{g}}$ (see [2]). If $\mathfrak{g}(p) \subset \mathfrak{g}$, then $\mathfrak{g}$-normalizers and $\mathfrak{g}$-hypercenter depend only on $\mathfrak{g}$ and not on $\mathfrak{g}(p)$ (see [1], [2]).

2. **Theorem.** Let $G$ be a soluble group, then

(a) Each $\mathfrak{g}$-normalizer of $G$ is not contained in any proper normal subgroup of $G$.

(b) The subgroup being generated by all $\mathfrak{g}$-normalizers of $G$ is $G$.

(c) The intersection of all $\mathfrak{g}$-normalizers of $G$ is the $\mathfrak{g}$-hypercenter.

This is a generalization of [3, Theorem VI, 11.11].

**Proof.** (a) If $M$ is a maximal subgroup of $G$, then $G/M$ is a central chief factor and so $G/M$ is $\mathfrak{g}$-central. Hence each $\mathfrak{g}$-normalizer $D$ of $G$ covers on $G/M$ by Theorem 4.1 of [1], so $D \subseteq M$.

(b) Since all $\mathfrak{g}$-normalizers of $G$ are conjugate, the subgroup being generated by all $\mathfrak{g}$-normalizers of $G$ is a normal subgroup containing an $\mathfrak{g}$-normalizer of $G$. From (a), we see it is $G$.

(c) As the hypercenter $Z_{\mathfrak{g}}$ of $G$ contains only $\mathfrak{g}$-central chief factors of $G$, so $Z_{\mathfrak{g}}$ is covered by every $\mathfrak{g}$-normalizer of $G$, thus $Z_{\mathfrak{g}} \subseteq \bigcap_{e \in \mathfrak{g}} D^e$. Set $\phi = \{T^e\}$ and $\phi^* = \{T^e Z_{\mathfrak{g}} / Z_{\mathfrak{g}}\}$. By [1, Theorem 4.1, Corollary 2], as $\mathfrak{g}$-normalizers are homomorphic invariant,

$$N_\mathfrak{g}(N(\phi)) Z_{\mathfrak{g}} / Z_{\mathfrak{g}} \subseteq N_\mathfrak{g}(Z_{\mathfrak{g}} / Z_{\mathfrak{g}}) = N_\mathfrak{g}(N(\phi^*)).$$

To prove the theorem, it is sufficient to show $\bigcap_{e \in \mathfrak{g}} N_\mathfrak{g}(N(\phi))^e = e$, because we see $\bigcap_{e \in \mathfrak{g}} N_\mathfrak{g}(N(\phi))^e Z_{\mathfrak{g}} / Z_{\mathfrak{g}} = e$ so $\bigcap_{e \in \mathfrak{g}} N_\mathfrak{g}(N(\phi))^e \subseteq Z_{\mathfrak{g}}$, since it is clear $\bigcap_{e \in \mathfrak{g}} N_\mathfrak{g}(\phi)^e \subseteq \bigcap_{e \in \mathfrak{g}} N_\mathfrak{g}(N(\phi))^e$, then $\bigcap_{e \in \mathfrak{g}} D^e = \bigcap_{e \in \mathfrak{g}} Z_{\mathfrak{g}}$, thus $\bigcap_{e \in \mathfrak{g}} D^e = Z_{\mathfrak{g}}$. Hence we may assume $Z_{\mathfrak{g}} = e$. Let $N$ be a minimal normal subgroup of $G$, which is contained in $N_\mathfrak{g}(D) = N_\mathfrak{g}(N(\phi))$. As $Z_{\mathfrak{g}} = e$, $N$ is an $\mathfrak{g}$-excentral, which is avoided by $N(\phi) = D$, by Theorem 4.1 of [1]. As $N \subseteq N_\mathfrak{g}(N(\phi))$ so that $[N, N(\phi)] \subseteq N \cap D = e$, $D$ is centralized by $N$, so $G$ generated by all $D$'s is centralized by $N$, which is a contradiction since $N$ would be in $Z_{\mathfrak{g}}$.

**References**


Saitama University, Urawa, Japan