ON $\mathfrak{G}$-NORMALIZERS AND $\mathfrak{G}$-HYPERCENTER

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Abstract. In this note, we shall prove a theorem which is a generalization of the following theorem: Let $G$ be a soluble group, then the intersection of all system normalizers of $G$ is the hypercenter of $G$.

Recently, Carter and Hawkes [1] have generalized the construction of system normalizers of finite soluble groups introducing the concept of the $\mathfrak{G}$-normalizer, and B. Huppert [2] has generalized the hypercenter introducing the concept of the $\mathfrak{G}$-hypercenter. In this note, we shall show the relations between $\mathfrak{G}$-normalizers and the $\mathfrak{G}$-hypercenter, which generalize the theorem [3, VI. 11.11] on the relations between the system normalizers and the hypercenter. The methods of proof of our theorem are similar to those in [3].

1. Definitions. All groups in this note are soluble and finite.

If nonempty formations $\mathfrak{F}(p)$, one for each $p$, are given, the local formation $\mathfrak{F}$ locally defined by $\{\mathfrak{F}(p)\}$ is the class of all groups $G$ such that, whenever $M$ is a chief factor of $G$, of order $p^n$, say, then the automorphisms group induced on $M$ by $G$ belongs to $\mathfrak{F}(p)$. And $M$ is called an $\mathfrak{F}$-central-$p$-chief factor. By Carter and Hawkes [1, p. 177] we can choose $\mathfrak{F}(p)$ such that $\mathfrak{F}(p) \subseteq \mathfrak{F}$. Let $\{S^p\}$ be a set of $p$-complements of $G$, one for each prime $p$ dividing $|G|$, and let $\gamma$ be the Sylow system of $G$ generated by $\{S^p\}$. We write $T^p = S^p \cap C^p$, where $C_p$ is the intersection of the centralizers of the $\mathfrak{F}$-central-$p$-chief factors of $G$. The set $T = \{T^p\}$ will be called an $\mathfrak{F}$-system of $G$ (see [1]). And then the subgroup $D = \cap_p N_\gamma(T^p)$ will be called the $\mathfrak{F}$-normalizer of $G$ (see [1]). Since any two Sylow systems of $G$ are conjugate in $G$, it follows that the $\mathfrak{F}$-normalizers of $G$ form a characteristic conjugacy class of subgroups of $G$. Let $N$ be a normal subgroup of $G$ with a normal chain $e = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = N$ where each $N_i$ is normal in $G$. $N$ is called an $\mathfrak{F}$-hypercentral subgroup whenever the following conditions are satisfied

1. $N_i/N_{i-1}$ is a chief factor of $G$.

2. If the order of $N_i/N_{i-1}$ is $p_i^n$, then $G/C_\gamma(N_i/N_{i-1}) \in \mathfrak{F}(p_i)$.

As a product of $\mathfrak{F}$-hypercentral normal subgroups is $\mathfrak{F}$-hypercentral, so the product of all $\mathfrak{F}$-hypercentral normal subgroups of $G$
is called the $\mathfrak{F}$-hypercenter of $G$, say $Z_{g}$ (see [2]). If $\mathfrak{F}(p) \subseteq \mathfrak{F}$, then $\mathfrak{F}$-normalizers and $\mathfrak{F}$-hypercenter depend only on $\mathfrak{F}$ and not on $\mathfrak{F}(p)$ (see [1], [2]).

2. Theorem. Let $G$ be a soluble group, then
(a) Each $\mathfrak{F}$-normalizer of $G$ is not contained in any proper normal subgroup of $G$.
(b) The subgroup being generated by all $\mathfrak{F}$-normalizers of $G$ is $G$.
(c) The intersection of all $\mathfrak{F}$-normalizers of $G$ is the $\mathfrak{F}$-hypercenter.

This is a generalization of [3, Theorem VI, 11.11].

Proof. (a) If $M$ is a maximal subgroup of $G$, then $G/M$ is a central chief factor and so $G/M$ is $\mathfrak{F}$-central. Hence each $\mathfrak{F}$-normalizer $D$ of $G$ covers on $G/M$ by Theorem 4.1 of [1], so $D \subseteq M$.

(b) Since all $\mathfrak{F}$-normalizers of $G$ are conjugate, the subgroup being generated by all $\mathfrak{F}$-normalizers of $G$ is a normal subgroup containing an $\mathfrak{F}$-normalizer of $G$. From (a), we see it is $G$.

(c) As the hypercenter $Z_{\mathfrak{F}}$ of $G$ contains only $\mathfrak{F}$-central chief factors of $G$, so $Z_{\mathfrak{F}}$ is covered by every $\mathfrak{F}$-normalizer of $G$, thus $Z_{\mathfrak{F}} \leq \bigcap_{\mathfrak{F}} D^{\mathfrak{F}}$. Set $\phi = \{ T^{\mathfrak{F}} \}$ and $\phi^{*} = \{ T^{\mathfrak{F}}Z_{\mathfrak{F}}/Z_{\mathfrak{F}} \}$. By [1, Theorem 4.1, Corollary 2], as $\mathfrak{F}$-normalizers are homomorphic invariant,

$$N_{\phi}(\phi^{*})Z_{\mathfrak{F}} / Z_{\mathfrak{F}} \subseteq N_{\phi/Z_{\mathfrak{F}}}(N(\phi)Z_{\mathfrak{F}} / Z_{\mathfrak{F}}) = N_{\phi/Z_{\mathfrak{F}}}(\phi^{*}).$$

To prove the theorem, it is sufficient to show $\bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} N_{\phi/Z_{\mathfrak{F}}}(N(\phi^{*}))^{\phi^{*}} = e$, because we see $\bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} N_{\phi}(N(\phi))^{\phi^{*}}Z_{\mathfrak{F}} / Z_{\mathfrak{F}} = e$ so $\bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} N_{\phi}(N(\phi))^{\phi^{*}} \subseteq Z_{\mathfrak{F}}$, since it is clear $\bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} N_{\phi}(\phi^{*})^{\phi} \subseteq \bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} N_{\phi}(N(\phi))^{\phi}$, then $\bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} D^{\phi^{*}} = \bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} N_{\phi}(\phi^{*})^{\phi} \subseteq Z_{\mathfrak{F}}$, thus $\bigcap_{\phi^{*} \in \phi/Z_{\mathfrak{F}}} D^{\phi^{*}} = Z_{\mathfrak{F}}$. Hence we may assume $Z_{\mathfrak{F}} = e$. Let $N$ be a minimal normal subgroup of $G$, which is contained in $N_{\phi}(D) = N_{\phi}(N(\phi))$. As $Z_{\mathfrak{F}} = e$, $N$ is an $\mathfrak{F}$-excentral, which is avoided by $N(\phi) = D$, by Theorem 4.1 of [1]. As $N \subseteq N_{\phi}(N(\phi))$ so that $[N, N(\phi)] \subseteq N \cap D = e$, $D$ is centralized by $N$, so $G$ generated by all $D$'s is centralized by $N$, which is a contradiction since $N$ would be in $Z_{\mathfrak{F}}$.

References


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