# ON THE FREE PRODUCT OF TWO GROUPS WITH AN AMALGAMATED SUBGROUP OF FINITE INDEX IN EACH FACTOR[1]

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ABSTRACT. Let $G=(A * B;\ U)$ where $U$ is finitely generated and of finite index $\neq 1$ in both $A$ and $B$. We prove that $G$ is a finite extension of a free group iff $A$ and $B$ are both finite. In particular, this answers in the negative a question of W. Magnus as to whether or not $G$ can be free. Analogous results are obtained for tree products and HNN groups.

1. **Introduction.** Let $G=(A * B;\ U)$ be a free product with an amalgamated subgroup, where $A$, $B$ are finitely generated groups. W. Magnus asked whether or not $G$ can ever be a free group if $U$ is of finite index ($\neq 1$) in both $A$ and in $B$. We prove that $G$ cannot be free in this case, and more generally that $G$ is a finite extension of a free group iff $A$ and $B$ are both finite groups. Moreover, we establish analogous results for a tree product and for an HNN group of the form

$$\langle t_1, \cdots, t_r, K; \text{rel } K, t_1L_1t_1^{-1} = M_1, \cdots, t_r^{-1}L_rt_r^{-1} = M_r\rangle$$

where $K$ is finitely generated and $L_i$, $M_i$ are each of finite index in $K$ (see [6] for definitions and notations).

2. **Lemmas.**

LEMMA 1. *If $G$ is a free group and $G=(A * B;\ U)$ where $U$ is finitely generated and of finite index in both $A$ and $B$, then $A=U$ or $B=U$.*

PROOF. Since $U$ is finitely generated, $U$ is a free factor of a subgroup $H$ which is of finite index in $G$ (see M. Hall Jr. [3] and [5]). It then follows that

$$A \cap H = U = B \cap H. \tag{1}$$

For, $A\cap H$ is a subgroup of $H$ and contains a free factor $U$ of $H$; hence $U$ is a free factor of $A\cap H$. But $U$ is of finite index in $A\cap H$; hence $A\cap H=U$, and similarly $B\cap H=U$.

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Since $H$ is of finite index in $G$, $H$ contains a subgroup $H_1$, normal and of finite index in $G$. Intersecting the equations (1) with $H_1$, we obtain

$$A \cap H_1 = U \cap H_1 = B \cap H_1. \tag{2}$$

Therefore, $U \cap H_1$ is a nontrivial normal subgroup of $A$ and of $B$, and hence normal in $G$. Consequently, $U$ is of finite index in $G$, which is impossible unless $A = U$ or $B = U$ (otherwise $(ab)^n$ where $a \in A - U$ and $b \in B - U$, $n = 0, \pm 1, \pm 2, \cdots$ determine infinitely many cosets of $U$ in $G$).

LEMMA 2. *Let $G = \Pi * (A_i; U_{jk} = U_{kj})$ be a tree product of groups $A_i$ with the subgroups $U_{jk}$ of $A_j$ and $U_{kj}$ of $A_k$ amalgamated. Suppose $G$ is a free group and each $U_{jk}$ is finitely generated and of finite index in $A_j$. Then $G$ equals one of its vertices $A_n$ and all the other vertices $A_i$ are of finite index in $A_n$.*

PROOF. We first consider the case where $G$ is the tree product of finitely many vertices $A_1, \cdots, A_r$ and use induction on $r$. Now the finite tree product $G$ has an extremal vertex say $A_r$, which is joined to a unique vertex, say $A_{r-1}$. The subgroup of $G$ generated by $A_1, \cdots, A_{r-1}$ is just their tree product. Hence by inductive hypothesis, each of these $r-1$ vertices is of finite index in one of them, say $A_1$. Then $G = (A_r * A_1; U_{r,r-1})$, and hence by Lemma 1, all vertices of $G$ are of finite index in either $A_1$ or in $A_r$.

Suppose now that $G$ has infinitely many vertices. Let $A_n$ be a vertex of minimum rank; moreover, if $A_n$ is a cyclic group, we may choose $A_n$ to be maximal among the vertices which are cyclic. We show that every other vertex $A_i$ of $G$ is of finite index in $A_n$. For, $A_i$ and $A_n$ are contained in a finite subtree of $G$; hence the vertices of this finite subtree are all of finite index in one of them, say $A_k$. From the Schreier rank formula, it follows that $A_k = A_n$.

## 3. The theorem for amalgamated products.

THEOREM 1. *Let $G = (A * B; U)$ where $U$ is finitely generated and of finite index ($\neq 1$) in both $A$ and $B$. Then $G$ has a free subgroup of finite index iff $A$ and $B$ are both finite groups.*

PROOF. If $A$ and $B$ are finite, then $G$ has a free subgroup of finite index (see the proof of Theorem 2 in G. Baumslag [1]).

Conversely, suppose $G$ has a free subgroup $H$ of finite index. Then by the subgroup theorem in [6], $H$ is an HNN group of the form

$$(3)\quad H = \langle t_1, t_2, \cdots, t_r, S; \operatorname{rel} S, t_1 U_H^{\delta_1} t_1^{-1} = U_H^{\delta_1'}, \cdots, t_r U_H^{\delta_r} t_r^{-1} = U_H^{\delta_r'} \rangle,$$

where $S$ is a tree product whose vertices are conjugates of $A$ or $B$ intersected with $H$, and whose amalgamated subgroups are conjugates of $U$ intersected with $H$. Moreover, neighboring vertices may be expressed in the form $A_H^D$, $B_H^D$ with $U_H^D$ as the subgroup amalgamated between them; also $A_H^{\delta_i}$ and $B_H^{\delta_i'}$ are among the vertices of $S$.

Since $U$ is finitely generated and of finite index in $A$ and $B$, the subgroup amalgamated between two vertices of $S$ is finitely generated and of finite index in both these vertices. Hence by Lemma 2, $S$ is equal to one of its vertices, and each of the vertices of $S$ is of finite index in $S$.

If $N$ is the normal subgroup of $H$ generated by $S$, then $N$ is itself a tree product in which the vertices are the conjugates of $S$ by the freely reduced words in $t_1, \cdots, t_r$; moreover, neighboring vertices have the form

$$W t_i^{\epsilon} S t_i^{-\epsilon} W^{-1}, \qquad W S W^{-1}$$

and the subgroup amalgamated between them is

$$W t_i^{\epsilon} U_H^{\alpha} t_i^{-\epsilon} W^{-1} = W U_H^{\beta} W^{-1}$$

where $\alpha = \delta_i$, $\beta = \delta_i'$ if $\epsilon = 1$, and $\alpha = \delta_i'$, $\beta = \delta_i$ if $\epsilon = -1$ (see Lemma 2 of [6]). Hence Lemma 2 applies to $N$, and we obtain that $N$ is one of its vertices, so that $N = S$ since $N$ is normal. But then $N$ is a finitely generated normal subgroup of the free group $H$, and so $N = 1$, or $N$ is of finite index in $H$ which by (3) implies $N = H$. However, $N \neq H$ because otherwise $H = S$, which by the above is a vertex $A_H^D$ or $B_H^D$; thus a conjugate of $H$ is in $A$ or in $B$, contrary to the fact that $A$ and $B$ are of infinite index in $G$. Therefore $N = 1$ and both $A$ and $B$ are finite since $A \cap H$, $B \cap H$ are both in $N$ and of finite index in $A$, $B$ respectively.

COROLLARY. *Let $G = \Pi * (A_i; U_{jk} = U_{kj})$ be a tree product of finitely many groups $A_i$ with the subgroups $U_{jk}$ of $A_j$ and $U_{kj}$ of $A_k$ amalgamated. Suppose each $U_{jk}$ is finitely generated and of finite index in $A_j$, and that for some $p$, $q$ we have $A_p \neq U_{pq}$ and $A_q \neq U_{qp}$. Then G has a free subgroup of finite index iff each $A_i$ is a finite group.*

PROOF. Suppose first that $G$ has a free subgroup of finite index; then $(A_p * A_q; U_{pq} = U_{qp})$ also has such. Hence by the above theorem $A_p$ must be finite. Moreover, since every vertex of $G$ can be joined to $A_p$ by a path each of whose edges is of finite index in the vertices of the edge, each $A_i$ must be finite.

Conversely, suppose each $A_i$ is finite. One shows by induction on the number of vertices of $G$ that there is a homomorphism of $G$ onto a finite group which is one-one on each $A_i$ and that a normal subgroup having trivial intersection with each $A_i$ is free.

## 4. An analogous theorem for HNN groups.

LEMMA 3. *Let $G$ be an HNN group*

$$G = \langle t_1, \cdots, t_r, K; \text{rel } K, t_1L_1t_1^{-1} = M_1, \cdots, t_rL_rM_r^{-1} = M_r\rangle. \tag{4}$$

*Then any subgroup $H$ of $G$ which has trivial intersection with each conjugate of $K$ must be a free group.*

PROOF. Let $X$ be the free group on $x_1, x_2, \cdots, x_r$, and $Y$ the free group on $y_1, y_2, \cdots, y_r$. Then

$$\begin{aligned} X * G &= Y * G \\ &= ((X * K) * (Y * K); K * x_1L_1x_1^{-1} * \cdots = K * y_1M_1y_1^{-1} * \cdots), \end{aligned}$$

where $t_i = y_i^{-1}x_i$ (see Higman, Neumann and Neumann [4] or [6]). We show that the conjugates of $H$ in $X * G$ have trivial intersection with $X * K$. Let $h$ ($\neq 1$) be an element of $H$, and let $w$ be in $X * G$. When $whw^{-1}$ is cyclically reduced as an element of $X * G$, an element $ghg^{-1}$ with $g \in G$ results. On the other hand if $whw^{-1} \in X * K$, then $whw^{-1}$ when cyclically reduced in $X*G$ must yield an element of $K$, contrary to $H$ having trivial intersection with the conjugates of $K$ in $G$. Similarly, $H$ has trivial intersection with the conjugates of $Y * K$ in $Y * G$. Therefore by a theorem of H. Neumann (see [8] or [6]), $H$ is free.

THEOREM 2. *Let $G$ be an HNN group as in* (4) *where $K$ is finitely generated and each $L_i$, $M_i$ is of finite index in $K$. Then $G$ has a free subgroup of finite index iff $K$ is finite. In particular, $G$ cannot be free.*

PROOF. Suppose first that $K$ is finite. Then $K$ can be embedded in a finite group $\Sigma$ (for example, the group of all permutations on the elements of $K$) in such a way that the conjugate of $L_i$ by some element $s_i$ in $\Sigma$ is $M_i$ (see, for example corollary on p. 57 in Carmichael [2], or Philip Hall's proof p. 537 in B. H. Neumann [7]). Map $G$ into $\Sigma$ by sending $K$ identically into itself in $\Sigma$ and mapping $t_i \to s_i$. Then $G$ is mapped homomorphically onto a finite group and the kernel is a normal subgroup having trivial intersection with $K$, and hence is free by Lemma 3.

Conversely, suppose that $G$ has a free subgroup of finite index and hence a normal free subgroup $H$ of finite index. Suppose $K$ is infinite. We first show that for each $i$, $L_i$ or $M_i$ must equal $K$. For otherwise the amalgamated product $(K * t_iKt_i^{-1}; M_i = t_iL_it_i^{-1})$ is a subgroup of $G$ and hence possesses a free subgroup of finite index, contrary to Theorem 1. Hence for each $i$, either $L_i$ or $M_i$ equals $K$. We may therefore assume (replacing $t_i$ by $t_i^{-1}$ if necessary) that $K = M_i$ for some $i$. Let $G_i = gp(t_i, K)$ and $H_i = G_i \cap H$. Then the groups $t_i^jKt_i^{-j} \cap H_i = t_i^j(K \cap H_i)t_i^{-j}$ form an ascending chain of free subgroups of the same finite rank in $H_i$. Hence for some $j$, $t_i^j(K \cap H_i)t_i^{-j} = t_i^{j+1}(K \cap H_i)t_i^{-j-1}$, so that $K \cap H_i = t_i(K \cap H_i)t_i^{-1}$. Therefore $K \cap H_i$ is normal in $G_i$ and therefore normal in $H_i$. But $K \cap H_i$ is finitely generated and is therefore of finite index in $H_i$ and hence of finite index in $G_i$, contrary to $K$ being of infinite index in $G_i$. Consequently $K$ must be a finite group.

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