A LOWER JACKSON BOUND ON $(-\infty, \infty)$

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ABSTRACT. We produce a lower bound for the degree of uniform polynomial approximation to continuous functions on the whole real line using the weight function $\exp(-|x|^{\alpha})$, $\alpha \geqq 2$. This lower bound has the same order of magnitude as the upper bound produced previously by Džrbašyan.

Given a continuous function $F(x)$ and a weight function $h(x)$ on $(-\infty, \infty)$ the modulus of continuity of F is defined by $\omega_F(\delta) = \operatorname{Sup} |F(x_1) - F(x_2)|$, the supremum being taken over all real x_1, x_2 satisfying $|x_1 - x_2| \leqq \delta(1+|x_1|)(1+|x_2|)$, and the Nth degree of approximation of F is given by

$$E_N(F, h) = \inf \operatorname{Sup} | F(x) - P(x) | h(x) = \inf \| F - P \|,$$

the infimum being taken over all polynomials $P(x)$ of degree $\leqq N$ and the supremum over all real x.

In an extension of the well-known result of Dunham Jackson [3], Džrbašyan [2] showed that, for $\alpha > 1$, $E_N(F, \exp(-|x|^{\alpha}))$ is bounded above by $C\omega_F(N^{-1+\beta})$, where $\beta = \alpha^{-1}$ and C is a positive constant. The following theorem shows that, for $\alpha \geqq 2$, this result is best possible.

THEOREM. *Let $\alpha \geqq 2$ and $\beta = \alpha^{-1}$. For each positive integer N there is a function F, continuous on $(-\infty, \infty)$, such that*

$$E_{2N-1}(F, \exp(-|x|^{\alpha})) > \frac{\sqrt{3}}{4}\, \omega_F(N^{-1+\beta}).$$

PROOF. We let $d\sigma(x)$ be the measure with masses $(-1)^k \binom{2N}{N+k}$ at the points $x = ka$, $k = 0, \pm 1, \cdots, \pm N$, $a = 4^{-\beta} N^{-1+\beta}$, so that

$$\int_{-\infty}^{\infty} x^n d\sigma(x) = \Delta^{2N} t^n \quad (t = -Na) = 0, \qquad n = 0, 1, \cdots, 2N-1.$$

If we now set $\sigma(x) = \int_{-\infty}^{x} d\sigma(t)$ and let $F(x)$ be the continuous "sawtooth" function satisfying $F(x) = 0$ for $|x| \geqq Na$ and $F'(x) = -\operatorname{sign} \sigma(x)$ for $|x| < Na$, $x \neq ka$, we see that $\omega_F(N^{-1+\beta}) = a$ and, for

Received by the editors August 16, 1969.

AMS 1968 *subject classifications.* Primary 4141, 4115.

Key words and phrases. Polynomial approximation on $(-\infty, \infty)$, degree of approximation on $(-\infty, \infty)$, Jackson's theorem on $(-\infty, \infty)$, weighted approximation on $(-\infty, \infty)$.

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any polynomial $P(x)$ of degree $\leqq 2N-1$,

$$\int_{-\infty}^{\infty} F(x)d\sigma(x) = \int_{-\infty}^{\infty} [F(x) - P(x)] \exp(-|x|^\alpha) \exp(|x|^\alpha) d\sigma(x).$$

Thus,

$$\begin{aligned}
\|F - P\|^{-1} \left| \int_{-\infty}^{\infty} F(x)d\sigma(x) \right| \\
&\leqq \sum_{k=-N}^{N} \binom{2N}{N+k} \exp(a^\alpha |k|^\alpha) \\
&\leqq \sum_{k=-N}^{N} \binom{2N}{N+k} \exp\left(\frac{k^2}{4N}\right) \\
&= \sum_{k=-N}^{N} \binom{2N}{N+k} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-x^2 + \frac{k}{\sqrt{N}} x\right) dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2) \left[\exp\left(\frac{x}{2\sqrt{N}}\right) + \exp\left(\frac{-x}{2\sqrt{N}}\right)\right]^{2N} dx \\
&< \frac{2}{\sqrt{3}} 4^N,
\end{aligned}$$

where the final inequality is obtained from the inequality $e^u + e^{-u} < 2\exp(\frac{1}{2}u^2)$ for $u \neq 0$.

However, integration by parts yields

$$\int_{-\infty}^{\infty} F(x)d\sigma(x) = -\int_{-\infty}^{\infty} F'(x)\sigma(x)dx = \int_{-\infty}^{\infty} |\sigma(x)| \, dx = \frac{1}{2} 4^N a$$

which, combined with the above, completes the proof.

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