GENERALIZED BALAYAGE AND A RADON-NIKODYM THEOREM

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Abstract. A simplified proof is given of Doob's result that a balayage ordered collection of probability measures on a compact Hausdorff space \( K \) yields a \( K \)-valued supermartingale with the measures as marginal distributions. The proof shows further connections with martingale convergence theory.

The main theorem of a recent article by Doob [1] says that a balayage ordered collection of probability measures on a compact Hausdorff space \( K \) yields a \( K \)-valued supermartingale with the given measures as marginal distributions. The theorem requires no metrizability assumptions, but its proof uses a theorem of Meyer on dilations of measures on a compact metrizable space [3, p. 246]. The purpose of this note is to give a simple proof of a version of Doob's theorem avoiding metrizability arguments. Meyer's theorem is an easy corollary. A Radon-Nikodym-Metivier type lemma makes use of the Ionescu Tulcea lifting theorem and Helms' martingale convergence theorem (see [3]), both of which are consequences of Doob's original martingale theory.

Let \( K \) be a compact Hausdorff topological space, let \( C(K) \) be the Banach lattice of continuous real valued functions on \( K \) with the supremum norm, and let \( C(K)^* \) be the Banach lattice dual of \( C(K) \). The space \( C(K)^* \) is identified with the space of all signed regular Borel measures on \( K \), and for \( f \) in \( C(K) \) and \( \lambda \) in \( C(K)^* \) the notation \( \lambda(f) = \langle \lambda, f \rangle = \int f d\lambda \) is used interchangeably. The set of positive elements of norm 1 in \( C(K)^* \) is denoted by \( P(K) \). A mapping \( T \) of \( K \) into \( C(K)^* \) is said to be weakly \( \lambda \)-measurable whenever the function \( x \rightarrow \langle T(x), f \rangle \) is \( \sigma \)-measurable for each \( f \) in \( C(K) \). The following lemma is a variation of a Radon-Nikodym theorem of Metivier [2].

Lemma. Suppose that \( \lambda \) is in \( P(K) \) and \( \mathcal{F} \) is the \( \sigma \)-field of \( \lambda \)-measurable sets. If \( m \) is an additive function from \( \mathcal{F} \) into the positive cone of \( C(K)^* \) such that \( |\langle m(E), f \rangle| \leq \|f\| \cdot \lambda(E) \) and \( \langle m(K), 1 \rangle = 1 \), then there is a

¹ The author would like to thank Professor J. L. Doob for several comments.
A weakly $\lambda$-measurable function $T$ from $K$ into $P(K)$ such that for each $E$ in $\mathcal{F}$ and for each $f$ in $C(K)$,

$$\langle m(E), f \rangle = \int_E \langle T(x), f \rangle \lambda(dx).$$

**Proof.** For each $f$ in $C(K)$, for each $x$ in $K$, and for each finite partition $\pi$ of $\mathcal{F}$ define $T(\pi, x, f) = \langle m(E), f \rangle / (\mathcal{E})$ where $\mathcal{E}$ is the member of $\pi$ containing $x$. If $\mathcal{F}_x$ denotes the Boolean algebra generated by $\pi$, the collection $\{ T(\pi, x, f), \mathcal{F}_x \}$ forms a bounded martingale for each $f$. By a theorem of Helms (see [3, p. 86]), the martingale converges in $L^1(K, \mathcal{F}, \lambda)$ to an integrable function $T$, which is clearly bounded by $||f||$. The mapping $f \mapsto T$ is a bounded linear mapping of $C(K)$ into $L^\infty(K, \mathcal{F}, \lambda)$. Let $\rho$ be a lifting of $L^\infty(K, \mathcal{F}, \lambda)$. The mapping $T$ defined by $\langle T(x), f \rangle = [\rho(T)](x)$, for each $f$ and $x$, is the desired function.

**Theorem.** Let $S$ be a cone in $C(K)$ closed under the lattice operation $\wedge$ and containing the constant functions. If $\mu$ and $\lambda$ are members of $P(K)$ such that $\mu(h) \leq \lambda(h)$ for each $h$ in $S$, then there is a weakly $\lambda$-measurable mapping $T$ from $K$ into $P(K)$ such that

$$\mu(f) = \int_K \langle T(x), f \rangle \lambda(dx) \quad \text{for } f \text{ in } C(K),$$

and

$$\int_E \langle T(x), h \rangle \lambda(dx) \leq \int_E h(x) \lambda(dx)$$

for each $h$ in $S$ and for each $\lambda$-measurable set $E$.

**Proof.** For each $f$ in $C(K)$ let $f^*(x) = \inf \{ h(x) : h \geq f, \ h \in S \}$. Properties of this function are given in [3]. Among these are the following: the equation $s(f) = \int f^*(x) \lambda(dx)$ defines a sublinear functional $s$ on $C(K)$, and since $\lambda$ dominates $\mu$ on $S$, $s$ dominates $\mu$ on $C(K)$. Let $B$ be the normed linear space of equivalence classes of $\lambda$-measurable simple functions from $K$ to $C(K)$. The space $C(K)$ can be identified with the $\lambda$-almost everywhere constant functions. The functional $s$ can be extended to a sublinear functional $s^*$ on $B$ defined by the equation $s^*(g) = \int g(x)^* \lambda(dx)$. By the Hahn-Banach theorem there is a linear functional $\mu^*$ dominated by $s^*$ which extends $\mu$ to all of $B$. For a subset $E$ of $K$ and for $f$ in $C(K)$ let $Ef$ denote the function whose values are $f$ on $E$ and 0 outside $E$. Thus $\mu^*(Ef) \leq s^*(Ef) = \int_{Ef} f^*(x) \lambda(dx)$ for $f$ in $C(K)$ and for $\lambda$-measurable $E$. 
The function $m$ defined by the equation $\langle m(E), f \rangle = \mu^*(Ef)$ satisfies the hypotheses of the lemma. The resulting function $T$ is the one required to conclude the proof.

The function $T$ considered as a transition probability on $K$ can be used to define a product measure $P$ on $K_1 \times K_2 = K \times K$. If $f(x, y)$ is a continuous function on $K \times K$ and $f_z(y) = f(x, y)$, then $P(f) = \int (T(x), f_z) \lambda(dx)$, so that $\lambda$ and $\mu$ are the natural projections of $P$ onto $K_1$ and $K_2$ respectively. Further, if $X_i$ is the $i$th coordinate function of a point $X$ in $K_1 \times K_2$ and $E$ is a Borel subset of $K$ then

$$\int_{E \times K} h(X_1) dP = \int_E h(x) \lambda(dx) \geq \int \langle T(x), h \rangle \lambda(dx) = \int_{B \times K} h(X_2) dP.$$  

Thus if $\mathcal{F}_2$ is the $\sigma$-field of Borel sets of $K \times K$ and $\mathcal{F}_1$ is composed of sets of the form $E \times K$ where $E$ is a Borel subset of $K$, then $\{(h(X_i), \mathcal{F}_i) : i = 1, 2\}$ is a supermartingale. This argument is easily extended to finite and then to arbitrary index sets to produce the following

**Corollary (Doob).** Suppose that $I$ is an ordered set and $\{\lambda_i : i \in I\}$ is a collection of members of $P(K)$ such that $i \leq j$ if and only if $\lambda_i(h) \geq \lambda_j(h)$ for each $h$ in $S$. There is a probability space $(\Omega, \mathcal{F}, P)$, a family $\{X_i : i \in I\}$ of $\mathcal{F}$-measurable functions, and a family $\{\mathcal{F}_i : i \in I\}$ of sub-$\sigma$-fields of $\mathcal{F}$ such that $\{(h(X_i), \mathcal{F}_i) : i \in I\}$ is a supermartingale for each $h$ in $S$ and $\lambda_i$ is the distribution of $X_i$ for each $i$ in $I$.

If $K$ is metrizable, then $T$ can be redefined as a Borel measurable function such that $\langle T(x), h \rangle \geq h(x)$ for each $x$ in $K$ and $h$ in $S$. This is Meyer's theorem.

**References**

