TRACE-CLASS FOR AN ARBITRARY H^*-ALGEBRA

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ABSTRACT. Let A be a proper H^*-algebra and let $\tau(A)$ be the set of all products xy of members x, y of A. Then $\tau(A)$ is a normed algebra with respect to some norm $\tau(\ )$ which is related to the norm $\| \ \|$ of A by the equality: $\|a\|^2 = \tau(a^*a)$, $a \in A$. There is a trace tr defined on $\tau(A)$ such that $\operatorname{tr}(a) = \sum_\alpha (ae_\alpha, e_\alpha)$ for each $a \in \tau(A)$ and each maximal family $\{e_\alpha\}$ of mutually orthogonal projections in A. The trace is related to the scalar product of A by the equality: $\operatorname{tr}(xy) = (x, y^*) = (y, x^*)$ for all $x, y \in A$.

1. The trace-class of operators (τc) was introduced by R. Schatten [5] as the set of all products of Hilbert-Schmidt operators acting on a Hilbert space. This class has its own norm, in which it is complete, and a trace, which can be used to define a scalar product on the set (σc) of all Hilbert-Schmidt operators to convert it into a simple H^*-algebra.

The present work deals with a generalization of this theory to an arbitrary H^*-algebra.

There are two ways this generalization can be achieved. One way would be to decompose an H^*-algebra into simple H^*-algebras, represent each simple H^*-algebra as a Hilbert-Schmidt class of operators, construct the corresponding trace-classes, take their direct sum and then derive the desired properties of the class $\tau(A) = \{xy \mid x, y \in A\}$ through identifying it with this direct sum. The other approach, subject of this paper, is to apply Schatten's technique directly to an H^*-algebra.

2. Let A be a proper H^*-algebra (A is a Banach algebra whose norm is a Hilbert space norm and which has an involution $x \to x^*$ such that $(y, x^*z) = (xy, z) = (x, zy^*)$ for all x, y, z in A (see [1])). A right centralizer on A is a bounded operator S on A such that $(Sx)y = S(xy)$ for all $x, y \in A$ [2]. Note that each operator of the form $La: x \to ax$ $(x, a \in A)$ is a right centralizer on A. A projection in A is a nonzero member e of A such that $e^2 = e = e^* \neq 0$ (e is a nonzero selfadjoint

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idempotent). For simplicity we shall refer to a maximal family $\{e_\alpha\}$ of mutually orthogonal projections as a *projection base* (note that in this case $A=\sum_\alpha e_\alpha A=\sum_\alpha Ae_\alpha$). A positive member of A is an element $a\in A$ such that $(ax, x)\geqq 0$ for all $x\in A$. Note that each positive member of A is selfadjoint. A normal element in A is some $b\in A$ such that $b^*b=bb^*$. A definition of the spectrum of a member of an algebra can be found on p. 28 of [**4**].

LEMMA 1, *Let b be a normal element in A. Then there exists a projection base $\{e_\alpha\}_{\alpha\in\Gamma}$ for A and a family $\{\lambda_\alpha\}_{\alpha\in\Gamma}$ of scalars such that $b=\sum_{\alpha\in\Gamma}\lambda_\alpha e_\alpha$. The nonzero numbers λ_α are nonzero numbers in the spectrum of b. If $b=a^*a$ for some $a\in A$ then every $\lambda_\alpha\geqq 0$.*

PROOF. Let B be a maximal commutative *-subalgebra of A containing b. Then B is a proper H^*-algebra: if $xB=0$ for some $x\in B$ then $x^*x=0$ and this implies that $x=0$ (Lemma 2.2, p. 370 in [**1**]). Then [**1**, Theorem 3.3] there exists a maximal family $\{e_\alpha\}_{\alpha\in\Gamma}$ of mutually orthogonal projections in B such that each $w\in B$ has the form $w=\sum_{\alpha\in\Gamma}\lambda_\alpha e_\alpha$. Note that each $\lambda_\alpha\neq 0$ is in the spectrum of w in B: if $\lambda_\alpha^{-1}w+y-\lambda_\alpha^{-1}wy=0$ for some $y\in B$ then

$$0=e_\alpha(\lambda_\alpha{}^{-1}w+y-\lambda_\alpha{}^{-1}wy)=\lambda_\alpha^{-1}e_\alpha w+e_\alpha y-\lambda_\alpha^{-1}e_\alpha wy$$
$$=e_\alpha+e_\alpha y-e_\alpha y=e_\alpha.$$

It follows that each $\lambda_\alpha\neq 0$ belongs to the spectrum of w in A also (see [**4**, p. 182]).

Note now that $\{e_\alpha\}$ is maximal relative to A also: if e is a projection orthogonal to each e_α then e is orthogonal to entire B, since the set of all linear combinations of members of $\{e_\alpha\}$ is dense in B; therefore $ex=xe$ for each $x\in B$ and so $e\in B$. But this would contradict maximality of $\{e_\alpha\}$:

If $w=a^*a$ for some $a\in A$ then $\lambda_\alpha\geqq 0$ for each α: $\lambda_\alpha(e_\alpha, e_\alpha)=(\lambda_\alpha e_\alpha, e_\alpha)$ $=(we_\alpha, e_\alpha)=(ae_\alpha, ae_\alpha)\geqq 0$ and $(e_\alpha, e_\alpha)\geqq 0$.

REMARK. Lemma 1 was originally stated (somewhat differently) for members of A of the form $b=a^*a$. The present form was suggested by the referee. We modified his proof somewhat.

COROLLARY 1. *For each $a\neq 0$ in A there exists a sequence $\{e_n\}$ of mutually orthogonal projections and a sequence $\{\lambda_n\}$ of positive numbers such that $a^*a=\sum_n\lambda_n e_n$. Note also that $a^*ae_n=e_na^*a=\lambda_ne_n$ for each n.*

Now for each n let $f_n=\lambda_n^{-1}ae_na^*$. Then $f_n^2=\lambda_n^{-2}ae_na^*ae_na^*=f_n$, $f_n^*=f_n$ and $f_nf_m=\lambda_n^{-1}\lambda_m^{-1}ae_na^*ae_ma^*=0$ if $m\neq n$. This simply means that $\{f_n\}$ is also a family of mutually orthogonal projections. Using this fact one can prove that the series $\sum_n \mu_n^{-1} e_n a^* x$, where $\mu_n = (\lambda_n)^{1/2}$ for each n, converges for each $x \in A$. It is done by showing that the series $\sum_{n=1}^{\infty} \|\mu_n^{-1} e_n a^* x\|^2$ converges:

$$\sum_{n=1}^{k} \|\mu_n^{-1} e_n a^* x\|^2 = \sum_{n=1}^{k} \mu_n^{-2}(e_n a^* x, e_n a^* x) = \sum_{n=1}^{k} (\lambda_n^{-1} a e_n a^* x, x)$$

$$= \sum_{n=1}^{k} (f_n x, x) = \sum_{n=1}^{k} \|f_n x\|^2 \leqq \|x\|^2$$

if k is any positive integer. Convergence of the series $\sum \mu_n^{-1} e_n a^* x$ will be used later.

Now let a be a fixed member of A and let $\{e_n\}$ and $\{\lambda_n\}$ be as in Corollary 1 ($a^*a = \sum_n \lambda_n e_n$). For each n let $\mu_n = (\lambda_n)^{1/2} \geqq 0$. Then

$$\sum_{n=1}^{k} \|\mu_n e_n\|^2 = \sum_{n=1}^{k} \mu_n^2(e_n, e_n) = \sum_{n=1}^{k} (\lambda_n e_n, e_n) = \sum_{n=1}^{k} (a^* a e_n, e_n)$$

$$= \sum_{n=1}^{k} \|a e_n\|^2 \leqq \|a\|^2$$

for each k and so $\sum_{n=1}^{\infty} \mu_n e_n$ converges. Define $[a] = \sum_n \mu_n e_n$. Then we have:

Lemma 2. *For each $a \in A$ there exists a unique positive member $[a]$ of A such that $[a]^2 = a^*a$ (note that $[a]^* = [a]$).*

Now for a given $a \in A$ we define a partial isometry W on A by setting $Wx = \sum_n \mu_n^{-1} a e_n x$ (where $\{\mu_n\}$ and $\{e_n\}$ are as above). Then W^* is also a partial isometry, $W^*x = \sum_n \mu_n^{-1} e_n a^* x$ for each $x \in A$ (convergence of this series was established above), $W[a] = a$, $W^*a = [a]$ and both W, W^* are right centralizers. Note also that $\|W\| = 1$, $\|W^*\| = 1$. We shall refer to the operator W as the partial isometry associated with a.

3. Now we follow the theory of R. Schatten developed on pp. 36–42 of [5].

Definition. We define the trace-class for A to be the set $\tau(A) = \{xy \mid x, y \in A\}$.

If $a \in \tau(A)$, $a = xy$ for some x, $y \in A$ and $\{e_\alpha\}$ is a projection base for A, then the series $\sum_\alpha (a e_\alpha, e_\alpha)$ converges absolutely, since

$$|(a e_\alpha, e_\alpha)| = |(y e_\alpha, x^* e_\alpha)| \leqq \|y e_\alpha\| \cdot \|x^* e_\alpha\| \leqq \tfrac{1}{2}(\|y e_\alpha\|^2 + \|x^* e_\alpha\|^2)$$

for each α (consult the proof of Lemma 4 on p. 30 of [5]). Note also that $\sum_\alpha |(a e_\alpha, e_\alpha)| < \frac{1}{2}(\|y\|^2 + \|x^*\|^2)$. We define tr $a = \sum_\alpha (a e_\alpha, e_\alpha)$

$=\sum_\alpha(ye_\alpha, x^*e_\alpha)=(y, x^*)$ and this expression is independent for both $\{e_\alpha\}$ and a particular decomposition of a into the product of x and y. Note also that $\operatorname{tr}(a^*a)=(a, a)=\|a\|^2$ for each $a\in A$.

LEMMA 3. *The following statements are equivalent*:

(i) $a\in\tau(A)$.

(ii) $[a]\in\tau(A)$.

(iii) *There exists a positive* $b\in A$ *such that* $b^2=[a]$.

(iv) $\sum_{\alpha\in\Gamma}([a]f_\alpha, f_\alpha)<\infty$ *for some projection base* $\{f_\alpha\}$.

PROOF. This lemma is the obvious modification of Lemma 2 on p. 37 of [5] and we need only to prove that (iv) implies (iii). Let $\{e_n\}$ and $\{\lambda_n\}$ be as in Corollary 1 above; for each n let $\gamma_n\geqq 0$ be such that $\gamma_n^4=\lambda_n$. Then $[a]=\sum_n \gamma_n^2 e_n$ and using this fact one can show that the series $\sum_n\|\gamma_n e_n x\|^2$ converges for each $x\in A$:

$$\sum_{n=1}^{k}\|\gamma_n e_n x\|^2=\left|\sum_{n=1}^{k}(\gamma_n^2 e_n x, x)\right|=\left|\sum_{n=1}^{k}(e_n[a]x, x)\right|$$
$$=\left|\left(\sum_{n=1}^{k}e_n[a]x, x\right)\right|\leqq\left\|\left(\sum_{n=1}^{k}e_n\right)[a]x\right\|\cdot\|x\|\leqq\|[a]\|\cdot\|x\|^2,$$

and this is valid for each k. We define the right centralizer T on A by setting $Tx=\sum_{n=1}^\infty\gamma_n e_n x$, $x\in A$. Then T is the positive square root of the operator $L[a]:x\to[a]x$, and this means that $\|Tf_\alpha\|^2=(T^2f_\alpha, f_\alpha)$ $=([a]f_\alpha, f_\alpha)$ for each f_α in the projection base in (iv). Statement (iv) in Lemma 3 then implies that the series $\sum_\alpha\|Tf_\alpha\|^2$ converges, i.e. there exists a countable subset $\Gamma_0=\{1, 2, \cdots, n, \cdots\}$ of Γ such that $Tf_\alpha=0$ if $\alpha\notin\Gamma_0$ and $\sum_{n=1}^\infty\|Tf_n\|^2<\infty$. We define $b=\sum_{\alpha\in\Gamma}Tf_\alpha$ $=\sum_{\alpha\in\Gamma_0}Tf_\alpha=\sum_{n=1}^\infty Tf_n$.[2] Let us show that $Tx=bx$ for each $x\in A$. Let $R=\sum_{n=1}^\infty f_nA$; then $R^p=\sum_{\alpha\notin\Gamma_0}f_\alpha A$ and $Tx=0=bx$ for each $x\in R^p$. If $x\in R$ then $x=\sum_{n=1}^\infty f_nx$ and

$$Tx=T\left(\lim_k\sum_{n=1}^{k}f_nx\right)=\lim_k T\left(\sum_{n=1}^{k}f_nx\right)=\lim_k\sum_{n=1}^{k}T(f_nx)$$
$$=\lim_k\sum_{n=1}^{k}(Tf_n)x=\lim_k\left(\sum_{n=1}^{k}Tf_n\right)x=\left(\lim_k\sum_{n=1}^{k}Tf_n\right)x$$
$$=\left(\sum_{n=1}^{\infty}Tf_n\right)x=bx.$$

It follows then that $b^2x=T^2x=[a]x$ for all $x\in A$, which implies that $b^2=[a]$, since A is proper.

[2] Note that the members of the family $\{Tf_\alpha\}$ are mutually orthogonal, since $Tf_\alpha=(Tf_\alpha)f_\alpha$ for each α.

As in the proof of Lemma 3, p. 38, of [5] we can show now that $\tau(A)$ is a linear subspace of A closed under the involution and right centralizers. It follows from Lemma 2.7, p. 372, in [1] that $\tau(A)$ is dense in A. It is now easy to see that tr is a positive linear functional on $\tau(A)$ such that $\operatorname{tr} a^* = \operatorname{tr} a^-$ and $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ for all $a \in \tau(A)$ and $x, y \in A$.

Now we define $\tau(a) = \operatorname{tr}[a]$. Then $\tau(a^*a) = \operatorname{tr}(a^*a) = \|a\|^2$ for all $a \in A$. To show that $\tau(\ )$ is a norm on $\tau(A)$ and to verify other properties of it we need the following two lemmas.

LEMMA 4. *If $a \in \tau(A)$ and S is a right centralizer then* $|\operatorname{tr}(S[a])| \leqq \|S\|\tau(a)$.

PROOF. Let b be a positive member of A such that $b^2 = [a]$. Then $|\operatorname{tr}(S[a])| = |\operatorname{tr}(Sb^2)| = |(Sb, b)| \leqq \|S\| \cdot \|b\|^2 = \|S\|\tau(b^*b) = \|S\|\tau(a)$.

COROLLARY 2. $|\operatorname{tr} a| \leqq \tau(a)$ *for each* $a \in \tau(A)$.

PROOF. Let W be a partial isometry associated with a. Then $|\operatorname{tr}(a)| = |\operatorname{tr}(W[a])| \leqq \|W\| \cdot \tau(a) = \tau(a)$.

LEMMA 5. *If a and S are as in Lemma 4 then* $\tau(Sa) \leqq \|S\|\tau(a)$.

PROOF. Let W_1, W_2 be partial isometries associated with Sa and a respectively (W_1, W_2 are right centralizers of norm 1 such that $W_1[Sa] = Sa$, $W_2[a] = a$). Then

$$\begin{aligned} \tau(Sa) &= \operatorname{tr}[Sa] = |\operatorname{tr}(W_1^*SW_2[a])| \leqq \|W_1^*SW_2\|\tau(a) \\ &\leqq \|W_1^*\| \cdot \|S\| \cdot \|W_2\| \cdot \tau(a) = \|S\|\tau(a). \end{aligned}$$

COROLLARY 3. *If $a \in \tau(A)$ then* $\|a\| \leqq \tau(a)$.

PROOF. First note that the operator $La^*: x \rightarrow a^*x$ is a right centralizer and that $\|La^*\| \leqq \|a^*\|$ (since $\|a^*x\| \leqq \|a^*\| \cdot \|x\|$ for each $x \in A$). Thus:

$$\|a\|^2 = \tau(a^*a) = \tau(La^*(a)) \leqq \|La^*\| \cdot \tau(a) \leqq \|a^*\| \cdot \tau(a) = \|a\|\tau(a);$$

hence $\|a\| \leqq \tau(a)$.

COROLLARY 4. *If $a, b \in A$ then $\tau(ab) \leqq \|a\| \cdot \|b\|$ and if $a, b \in \tau(A)$ then $\tau(ab) \leqq \tau(a)\tau(b)$.*

PROOF. Let W be a partial isometry associated with ab. Then

$$\begin{aligned} \tau(ab) &= \operatorname{tr}[ab] = \operatorname{tr}(W^*ab) = (W^*a, b^*) \\ &\leqq \|W^*\| \cdot \|a\| \cdot \|b^*\| = \|a\| \cdot \|b\| \leqq \tau(a) \cdot \tau(b). \end{aligned}$$

4. Now we can state our main result.

THEOREM. *Let A be a proper H^*-algebra and let $\tau(A)$ be the set of all products xy of members x, y of A. Then $\tau(A)$ coincides with the set of those members a of A for which the series $\sum_\alpha([a]e_\alpha, e_\alpha)$ converges for some projection base $\{e_\alpha\}$ of A. If $a\in\tau(A)$ then this series converges (absolutely) for each projection base $\{e_\alpha\}$ and the sum $\tau(a)=\sum_\alpha([a]e_\alpha, e_\alpha)$ does not depend on a particular choice of $\{e_\alpha\}$. There is a positive linear functional* tr *defined on $\tau(A)$ such that* $\mathrm{tr}(xy)=(y, x^*)=(x, y^*)=\mathrm{tr}(yx)$ *for all $x, y\in A$ and* $\mathrm{tr}\, a=\sum_\alpha(ae_\alpha, e_\alpha)$ *for each $a\in\tau(A)$ and each projection base $\{e_\alpha\}$. The trace-class $\tau(A)$ is a two-sided ideal in A, dense in A and closed under the involution and right centralizers. In fact $\tau(A)$ is a normed algebra with respect to the norm* $\tau(a)=\mathrm{tr}[a]=\sum_\alpha([a]e_\alpha, e_\alpha)$, *which has the property that* $|\mathrm{tr}\, a|\leqq\tau(a)$, $\|a\|\leqq\tau(a)$ *and* $\tau(ab)\leqq\tau(a)\tau(b)$ *for all $a, b\in\tau(A)$.*

In the next paper it will be shown that $\tau(A)$ is complete, i.e. $\tau(A)$ is a Banach algebra.

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