

## TRACE-CLASS AND CENTRALIZERS OF AN $H^*$ -ALGEBRA<sup>1</sup>

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ABSTRACT. Let  $A$  be a proper  $H^*$ -algebra. Let  $\tau(A) = \{xy \mid x, y \in A\}$ , let  $R(A)$  be the set of all bounded linear operators  $S$  on  $A$  such that  $S(xy) = (Sx)y$  for all  $x, y \in A$  and let  $C(A)$  be the closed subspace of  $R(A)$  generated by the operators of the form  $La: x \rightarrow ax, a \in A$ . It is shown that  $\tau(A)$  can be identified with the space of all bounded linear functionals on  $C(A)$  and that  $R(A)$  is the dual of  $\tau(A)$ . Also it is proved that  $\tau(A)$  is a Banach algebra.

This paper is a continuation of [4]. It generalizes Theorems 2, 3 on pp. 47, 48 of [6]. These theorems state that the trace-class ( $\tau c$ ) of operators on a Hilbert space  $H$  can be identified with the set of all bounded linear functionals on the ideal of completely continuous operators on  $H$  and that the dual of ( $\tau c$ ) is essentially the set of all bounded operators on  $H$ .

A consequence of the generalization of the first fact is the completeness of the trace-class  $\tau(A)$  for an arbitrary proper  $H^*$ -algebra  $A$ .

Let  $A$  be a proper  $H^*$ -algebra [1]. We define  $R(A)$  to be the set of all right centralizers on  $A$  [2] ( $R(A)$  is the set of all bounded operators  $S$  on  $A$  such that  $S(xy) = (Sx)y$  for all  $x, y \in A$ ). We define  $C(A)$  to be the closed subspace of  $R(A)$  generated by the operators  $La: x \rightarrow ax, a \in A$ , acting on  $A$ ;  $C(A)$  is the closure in the operator norm of the set  $\{La \mid a \in A\}$ .

For each  $a \in \tau(A)$  consider the mapping  $f_a$  on  $C(A)$  defined by  $f_a(S) = \text{tr}(Sa), S \in C(A)$ . Then  $|f_a(S)| = |\text{tr}(Sa)| \leq \tau(Sa) \leq \|S\| \tau(a)$  for each  $S \in C(A)$  [4, Lemma 5] and so  $f_a$  is a bounded linear functional on  $C(A)$  with  $\|f_a\| \leq \tau(a)$ . The converse inequality also holds:

LEMMA. *If  $a \in \tau(A)$  then the mapping  $f_a: S \rightarrow \text{tr}(Sa)$ , defined on  $C(A)$ , is a bounded linear functional and  $\|f_a\| = \tau(a)$ .*

PROOF. We only need to prove that  $\tau(a) \leq \|f_a\|$ . Let  $\{e_n\}$  and  $\{\mu_n\}$  be respectively a sequence of mutually orthogonal projections and a sequence of positive numbers such that  $[a] = \sum_{n=1}^{\infty} \mu_n e_n$  and

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$[a]e_k = e_k[a] = \mu_k e_k$  for each  $k$  (see the definition of  $[a]$  in [4] (before Lemma 2)).

For each  $n$  let  $c_n = \sum_{k=1}^n e_k$ ; then  $Lc_n \in C(A)$ ,  $\|Lc_n\| = 1$  and

$$\begin{aligned} f[a](Lc_n) &= \text{tr}(Lc_n([a])) = \text{tr}(c_n[a]) = \text{tr}\left(\sum_{k=1}^n e_k[a]\right) \\ &= \text{tr}\left(\sum_{k=1}^n \mu_k e_k\right) = \sum_{k=1}^n \mu_k \text{tr}(e_k) = \sum_{k=1}^n \mu_k(e_k, e_k). \end{aligned}$$

Thus we have:

$$\sum_{k=1}^n \mu_k(e_k, e_k) = |f[a](Lc_n)| \leq \|f[a]\| \cdot \|Lc_n\| = \|f[a]\|.$$

It follows that

$$\tau(a) = \text{tr}[a] = \sum_{k=1}^{\infty} ([a]e_k, e_k) = \sum_{k=1}^{\infty} \mu_k(e_k, e_k) \leq \|f[a]\|,$$

and so we only have to show now that  $\|f[a]\| \leq \|f_a\|$ .

Let  $W$  be the partial isometry associated with  $a$  (see definition of  $W$  in [4] after Lemma 2). Then  $W, W^*$  are right centralizers of norm 1,  $W[a] = a, W^*a = [a]$  and we have for each  $S \in C(A)$ :

$$\begin{aligned} |f[a](S)| &= |\text{tr}(S[a])| = |\text{tr}(SW^*a)| = |f_a(SW^*)| \\ &\leq \|f_a\| \cdot \|SW^*\| \leq \|f_a\| \cdot \|S\| \cdot \|W^*\| \leq \|f_a\| \cdot \|S\|. \end{aligned}$$

This simply means that  $\|f[a]\| \leq \|f_a\| \dots$

**THEOREM 1.** *Each bounded linear functional on  $C(A)$  is of the form  $f_a$  for some  $a \in \tau(A)$ . The correspondence  $a \rightarrow f_a$  is an isometric isomorphism between  $\tau(A)$  and  $C(A)^*$ .*

**PROOF.** The fact that the mapping  $a \rightarrow f_a$  is one-to-one is easy to verify: if  $f_a = f_b$  then the equality  $(a, x^*) = \text{tr}(xa) = \text{tr}(Lx(a)) = f_a(Lx) = f_b(Lx) = \text{tr}(xb) = (b, x^*)$  holds for each  $x \in A$ , and this means that  $a = b$ .

To prove that the mapping  $a \rightarrow f_a$  is onto, we use Riesz' Theorem about linear functionals on a Hilbert space. Let  $f$  be a bounded linear functional on  $C(A)$ . Then the mapping  $x \rightarrow f(Lx)$  is a bounded linear functional on  $A$  since  $|f(Lx)| \leq \|f\| \cdot \|Lx\| \leq \|f\| \cdot \|x\|$  (note that  $\|Lx\| \leq \|x\|$  for each  $x \in A$ ). Hence there exists  $a \in A$  such that  $f(Lx) = (x, a^*)$  for all  $x \in A$ . Then  $f(Lx) = (x, a^*) = \text{tr}(ax) = \text{tr}(xa) = \text{tr}(Lx(a))$  for each  $x \in A$ . If we could show that  $a \in \tau(A)$  this would imply that  $f$  is of the form  $f = f_a$ .

So let  $\{e_n\}$ ,  $\{\mu_n\}$  and  $W$  be as in the above lemma ( $a^*a = \sum_n \mu_n^2 e_n$ ,  $[a] = \sum_n \mu_n e_n$ ,  $W[a] = a$ ,  $W^*a = [a]$  and  $Wx = \sum_n \mu_n^{-1} a e_n x$ ,  $W^*x = \sum_n \mu_n^{-1} e_n a^* x$  for each  $x \in A$ ) (see discussion following Lemma 2 in [4]). For each  $n$  let  $c_n = \sum_{k=1}^n e_k$ ,  $d_n = \sum_{k=1}^n \mu_k^{-1} e_k a^*$ . Then

$$Ld_n(x) = d_n x = \sum_{k=1}^n \mu_k^{-1} e_k a^* x = c_n \sum_{k=1}^{\infty} \mu_k^{-1} e_k a^* x = Lc_n(W^*x)$$

and so  $\|Ld_n(x)\| = \|Lc_n(W^*x)\| \leq \|Lc_n\| \cdot \|W^*\| \cdot \|x\| \leq \|x\|$ , which means that  $\|Ld_n\| \leq 1$ . Then we have for each  $n$ :

$$\begin{aligned} \sum_{k=1}^n \mu_k(e_k, e_k) &= \sum_{k=1}^n \mu_k \operatorname{tr}(e_k) = \left| \operatorname{tr} \left( \sum_{k=1}^n \mu_k e_k \right) \right| = \left| \operatorname{tr} \left( \sum_{k=1}^n \mu_k^{-1} e_k a^* a \right) \right| \\ &= \left| \operatorname{tr}(d_n a) \right| = \left| \operatorname{tr}(Ld_n(a)) \right| = \left| f(Ld_n) \right| \\ &\leq \|f\| \cdot \|Ld_n\| = \|f\|. \end{aligned}$$

This simply means that the series  $\sum_{k=1}^{\infty} \mu_k(e_k, e_k)$  converges,  $b = \sum_k (\mu_k)^{1/2} e_k$  belong to  $A$  and  $[a] = b^2$  belongs to  $\tau(\quad)$ .

**COROLLARY.**  $\tau(A)$  is a Banach algebra in the norm  $\tau(A)$ .

**PROOF.**  $\tau(A)$  is complete since it is isometric to the dual of  $C(A)$  (see 8A in [3]).

**THEOREM 2.** For each  $S \in R(A)$  the mapping  $f_S: x \rightarrow \operatorname{tr}(Sx)$ ,  $f_S(\dot{x}) = \operatorname{tr}(Sx)$ , is a bounded linear functional on  $\tau(A)$  such that  $\|f_S\| = \|S\|$ . Conversely, each bounded linear functional on  $\tau(A)$  is of the form  $f_S$  for some  $S \in R(A)$ . Thus  $R(A)$  is isomorphic and isometric to the space  $\tau(A)^*$  of bounded linear functionals on  $\tau(A)$ .

**PROOF.** Inequality  $|\operatorname{tr}(Sx)| \leq \|S\| \cdot \tau(x)$  (Lemma 5 in [4]) implies that  $f_S$  is a bounded linear functional on  $\tau(A)$  and that  $\|f_S\| \leq \|S\|$ . To establish the converse inequality we select for a given  $\epsilon > 0$  a member  $a \in A$  with  $\|a\| = 1$  such that  $\|S^*\|^2 - \epsilon \leq \|S^*a\|^2$  and then consider:

$$\begin{aligned} \|S^*\|^2 - \epsilon &\leq \|S^*a\|^2 = (S^*a, S^*a) = |(SS^*a, a)| = |\operatorname{tr}(SS^*(aa^*))| \\ &= |\operatorname{tr}(S(S^*aa^*))| = |f_S(S^*aa^*)| \\ &\leq \|f_S\| \cdot \tau(S^*(aa^*)) \leq \|f_S\| \cdot \|S^*\| \cdot \tau(aa^*) \\ &= \|f_S\| \cdot \|S^*\| \operatorname{tr}(aa^*) \\ &= \|f_S\| \cdot \|S^*\| \cdot \|a\|^2 = \|f_S\| \cdot \|S^*\|. \end{aligned}$$

Since this is valid for each  $\epsilon > 0$  we have:  $\|S\| = \|S^*\| \leq \|f_S\|$ .

Now let  $f \in \tau(A)^*$ . For a fixed  $a \in A$  the linear functional  $g_a: x \rightarrow f(ax)$

is bounded on  $A$ :  $|g_a(x)| = |f(ax)| \leq \|f\| \tau(ax) \leq \|f\| \cdot \|a\| \cdot \|x\|$  (Corollary 4 in [4]) and so  $\|g_a\| \leq \|f\| \cdot \|a\|$ . Hence there is a member  $b_a$  of  $A$  such that  $f(ax) = g_a(x) = (x, b_a^*) = \text{tr}(b_a x)$  for each  $x \in A$  and  $\|b_a\| \leq \|f\| \cdot \|a\|$ . Define the mapping  $S$  on  $A$  by setting  $Sa = b_a$  for each  $a \in A$ . Then  $S$  is a bounded linear operator on  $S$  such that  $\|S\| \leq \|f\|$  and  $\text{tr}((Sa)x) = f(ax)$  for all  $x, a \in A$ . Let us show that  $S$  is a right centralizer. If  $u, v \in A$  then we have for each  $x \in A$ :

$$\begin{aligned} (S(uv), x^*) &= \text{tr}(xS(uv)) = \text{tr}((S(uv))x) = f((uv)x) = f(u(vx)) \\ &= \text{tr}(Su(vx)) = (Su, (vx)^*) = (Su, x^*v^*) = ((Su)v, x^*). \end{aligned}$$

Thus  $S(uv) = (Su)v$ . . . .

It follows also that  $f = f_S$ : if  $a = uv$  for some  $u, v \in A$  then  $f(a) = f(uv) = \text{tr}((Su)v) = \text{tr}(S(uv)) = \text{tr}(Sa) = f_S(a)$ .

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