ON THE COHOMOLOGY CHERN CLASSES OF THE K-THEORY CHERN CLASSES

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ABSTRACT. Let ξ be a vector bundle over a finite complex and $\gamma^i\xi$ its ith-$\boldsymbol{K}$ theory Chern class. We first show that

$$c_n\gamma^i\xi = (i-1)!S(n,i)c_n\xi + \text{decomposables},$$

where $S(n, i)$ is a Stirling number of the second kind. We apply this result to show that certain multiples of the e-invariant of a map $S^{2m-1}\to S^{2n}$ must always be integral.

1. **On Chern classes of Chern classes.** Let X be a finite complex and $\xi\in\tilde{\boldsymbol{K}}(X)$. Then ξ has K-theory Chern classes [6] $\gamma^i\xi\in\boldsymbol{K}^{2i}(X)$ which in turn have cohomology Chern classes $c_n\gamma^i\xi\in H^{2n}(X;\boldsymbol{Z})$.

Our objective is to study the problem of computing $c_n\gamma^i\xi$ as a function of $c_j\xi$. Contemplation of the universal example shows that there must be a universal formula expressing $c_n\gamma^i\xi$ as a polynomial in $c_0\xi$, $\cdots$, $c_n\xi$. As a step towards computing this polynomial we shall establish:

THEOREM 1.1. *Let X be a finite complex and ξ a vector bundle over X. Then we have*

$$c_n\gamma^i\xi = (i-1)!S(n,i)c_n(\xi) + \textit{decomposables}.$$

(Here $S(n,i)$ is a Stirling number of the second kind. See the appendix for information about Stirling numbers.)

PROOF. It will clearly suffice to check the assertion in the universal example, i.e., with X a giant skeleton of $\boldsymbol{BU}$ and ξ the restriction of the canonical bundle. Let

$$\phi_n: S^{2n}\to \boldsymbol{BU}$$

be a representative of a generator of $\pi_{2n}(BU)\cong\boldsymbol{Z}$. Recall [2] that

$$c_n(\xi_n) = (n-1)!\sigma_n$$

where ξ_n is the bundle induced by ϕ_n over S^{2n} and $\sigma_n\in H^{2n}(S^{2n};\boldsymbol{Z})$ is the canonical generator.

Now recall that we have shown [5, 2.4, 2.5] that

$$\mathrm{ch}^{2n}\gamma^i\xi_n = (i-1)!S(n,i)\sigma_n.$$

Received by the editors December 30, 1969.

AMS 1969 *subject classifications*. Primary 5732, 5540; Secondary 5530, 1005.

Key words and phrases. K-theory, characteristic classes, Chern classes, e-invariant, Stirling numbers.

However we also have the universal formula

$$\mathrm{ch}^{2n}\,\eta = (1/n!)(nc_n(\eta) + \text{decomposables})$$

for any bundle η over a finite complex. Since decomposables vanish in a sphere this specializes to

$$\mathrm{ch}^{2n}\,\gamma^i\xi_n = (1/(n-1)!)c_n\gamma^i\xi_n.$$

Equating these two expressions for $\mathrm{ch}^{2n}\gamma^i\xi_n$ leads to the formula

$$c_n\gamma^i\xi_n = (i-1)!S(n,i)(n-1)!\sigma_n$$

and recalling that $(n-1)!\sigma_n = c_n\xi_n$ we obtain

$$c_n\gamma^i\xi_n = (i-1)!\,S(n,i)c_n\xi_n.$$

If we let θ denote the universal bundle over $\boldsymbol{BU}$ then we have the universal formula $c_n\gamma^i\theta = A_{n,i}c_n\theta + \text{decomposables}$. Applying ϕ_n^* gives $c_n\gamma^i\xi_n = A_{n,i}c_n\xi_n$ and the result follows.□

Corollary 1.2. *Let X be a finite complex and ξ a vector bundle over X. Then we have*

$$\mathrm{ch}^{2n}\,\gamma^i\xi = (i-1)!S(n,i)\mathrm{ch}^{2n}\xi + \textit{decomposables}.$$

Proof. This is implicit in the proof of (1.1) above and in addition follows easily from it.□

Corollary 1.3. *Let X be a finite complex and ξ a vector bundle over X. Suppose that all products are zero in $\tilde{H}^*(X;\boldsymbol{Q})$, for example X is a suspension. Then*

$$\mathrm{ch}^{2n}\,\gamma^i\xi = (i-1)!S(n,i)\;\mathrm{ch}^{2n}\,\xi.$$

If moreover all products are zero in $\tilde{H}^*(X;\boldsymbol{Z})$ then

$$c_n\gamma^i\xi = (i-1)!S(n,i)c_n\xi. \quad \square$$

2. **On the unstable e-invariant.** Suppose given $f: S^{2m-1}\to S^{2n}$. Form the cofibration

$$S^{2n} \underset{i}{\hookrightarrow} S^{2n}\cup_f e^{2m} \underset{j}{\to} S^{2m}$$

yielding the exact sequence

$$0 \leftarrow \tilde{K}(S^{2n}) \overset{i^*}{\leftarrow} \tilde{K}(S^{2n}\cup_f e^{2m}) \overset{j^*}{\leftarrow} \tilde{K}(S^{2m}) \leftarrow 0.$$

Choose generators λ, $\mu \in K(S^{2n}\cup_f e^{2m})$ such that $j^*\xi_m = \lambda$, $i^*\mu = \xi_n$, where for any integer k, $\xi_k \in \tilde{K}(S^{2k}) \cong \boldsymbol{Z}$ is the canonical generator. We then have

$$\operatorname{ch}\lambda = \sigma_m, \qquad \operatorname{ch}\mu = e(f)\sigma_m + \sigma_n: \quad e(f) \in \boldsymbol{Q},$$

where as previously $\sigma_k \in H^{2k}(S^{2k}; \boldsymbol{Z}) \cong \boldsymbol{Z}$ is the canonical generator. The rational number $e(f)$ is of course the e-invariant of f and its residue class in $\boldsymbol{Q}/\boldsymbol{Z}$ is independent of the choice of λ, μ [1], [7]. Our objective is to show that certain multiples of $e(f)$ must always be integral. To this end we introduce:

DEFINITION. Let $m, n, k \in \boldsymbol{Z}^+$, and set

$$g(m, n) = \underset{k>1}{\text{g.c.d.}}\{k^m - k^n\}.$$

In the appendix we will establish that

$$g(m, n) = \underset{1<k<1+\max(m+1,n+1)}{\text{g.c.d.}} \{(k-1)![S(m+1, k) - S(n+1, k)]\}$$

where $m, n, k \in \boldsymbol{Z}^+$.

The main result of this section is the following.

THEOREM 2.1. *Let $f: S^{2m-1} \to S^{2n}$ and assume that $m \neq 2n$. Then $g(m-1, n-1)e(f) \in \boldsymbol{Z}$.*

This result is among results originally obtained by P. Hoffman in [3] extending a stable result of Adams [1] and Toda [7].

PROOF OF 2.1. Let us continue to employ the notations of the previous discussion. Then on general grounds we have

$$\text{(A)} \qquad \gamma^i\mu = r_i\mu + s_i\lambda: \qquad r_i, s_i \in \boldsymbol{Z}.$$

Moreover as we are assuming $m \neq 2n$ we obtain in view of (1.3)

$$\text{(B)} \qquad \operatorname{ch}^{2m}\gamma^i\mu = (i-1)!S(m, i)\operatorname{ch}^{2m}\mu = (i-1)!S(m, i)e(f)\sigma_m,$$

$$\text{(C)} \qquad \operatorname{ch}^{2n}\gamma^i\mu = (i-1)!S(n, i)\operatorname{ch}^{2n}\mu = (i-1)!S(n, i)\sigma_n,$$

while from (A) we obtain

$$\text{(D)} \qquad \operatorname{ch}^{2m}\gamma^i\mu = r_i\operatorname{ch}^{2m}\mu + s_i\operatorname{ch}^{2m}\lambda = r_ie(f)\sigma_m + s_i\sigma_m,$$

$$\text{(E)} \qquad \operatorname{ch}^{2n}\gamma^i\mu = r_i\operatorname{ch}^{2n}\mu + s_i\operatorname{ch}^{2n}\lambda = r_i\sigma_n.$$

Equating (C) and (E) leads to

$$\text{(F)} \qquad r_i = (i-1)!S(n, i),$$

while equating (B) and (D) and employing (F) gives

$$\text{(G)} \qquad (i-1)![S(m, i) - S(n, i)]e(f) = s_i \in \boldsymbol{Z}.$$

Since i is arbitrary we conclude that

$$\text{g.c.d.}\{(i-1)![S(m, i) - S(n, i)]\}e(f) \in \boldsymbol{Z}.$$

But $g(m-1, n-1)=\text{g.c.d.}\{(i-1)![S(m, i)-S(n, i)]\}$ by the theorem from the appendix previously quoted. □

Appendix: On Stirling numbers. Let us first recall one definition of the Stirling numbers of the second kind [4, II.7].

DEFINITION. Let n, k be positive integers. Then

$$S(n, k) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n.$$

Note that

$$S(n, 1) = 1 = S(n, n), \qquad S(n, k) = 0: \quad k > n,$$

and

$$S(n+1, k) = S(n, k-1) + kS(n, k),$$

and so from the two conditions above we obtain $S(n, k) \in \boldsymbol{Z}^+$; $k = 1, \cdots, n$.

These numbers have a long history and many useful properties, see for example [4], [5].

The formula

$$(-1)^k k! S(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n$$

may be inverted to yield a formula expressing k^n as a function of $S(n, j)$, $j=1, \cdots, n$. (This formula is often used to define the Stirling numbers.) The procedure runs as follows.

LEMMA 1. *Let $m, n \in \boldsymbol{Z}^+$, then*

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} (-1)^n \binom{j}{n} = \delta_{mn}$$

whenever $n \leqq m$.

PROOF. Start from the identity $t^m = (1-(1-t))^m$ and apply the binomial theorem twice to yield

$$t^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1-t)^j = \sum_{j,n=0}^{m} (-1)^j \binom{m}{j} (-1)^n \binom{j}{n} t^n$$

and the result follows from equating coefficients. □

Return now to the formula

$$(-1)^k k! S(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n.$$

Multiplying by $(-1)^k\binom{i}{k}$ and summing from $k=0$ to $k=i$ we get

$$\sum_{k=0}^{i} k!\binom{i}{k}S(n,k) = \sum_{k=0}^{i}\sum_{j=0}^{k}(-1)^k\binom{i}{k}(-1)^j\binom{k}{j}j^n.$$

Interchanging the order of summation and applying Lemma 1 we obtain

$$\sum_{k=0}^{i} k!\binom{i}{k}S(n,k) = \sum_{j=0}^{k}\delta_{ij}j^n = i^n.$$

Hence we have established:

PROPOSITION 2. *We have the formula* $i^n = \sum_{x=0}^{i} k!\binom{i}{k}S(n,k)$, *for all* $n, i \in Z^+$. □

COROLLARY 3. *Let* n, m, k *be positive integers. Then we have the formulas*

$$k^m - k^n = \sum_{i=0}^{k} i!\binom{k}{i}[S(m,i) - S(n,i)]$$

and

$$(-1)^k k![S(m,k) - S(n,k)] = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(i^m - i^n).$$

PROOF. The first assertion follows from Proposition 2 and the second from the definitions. □

COROLLARY 4. *Let* $n, m \in Z^+$, *then*

$$g(m,n) = \underset{1<k<1+\max(n,m)}{\text{g.c.d.}}\{k![S(m,k) - S(n,k)]\}.$$

PROOF. Immediate from Corollary 3 and the fact that $S(r,s)=0$ for $s>r$. □

More generally than Corollary 4 we have:

PROPOSITION 5. *Let* $m, n \in Z^+$. *Then*

$$(-1)^k(k-1)![S(m,k) - S(n,k)] = \sum_{i=1}^{k}(-1)^i\binom{k-1}{i-1}(i^{m-1} - i^{n-1}),$$

$$k \in Z^+.$$

PROOF. The result takes almost as long to state as to prove. Start with the formula

$$(-1)^k k![S(m,k) - S(n,k)] = \sum (-1)^i \binom{k}{i} (i^m - i^n).$$

Using the binomial identity

$$\binom{k}{i}\frac{1}{k} = \binom{k-1}{i-1}\frac{1}{i}$$

rewrite it as

$$(-1)^k (k-1)![S(m,k) - S(n,k)] = \sum (-1)^i \binom{k-1}{i-1}\frac{1}{i}(i^m - i^n)$$

$$= \sum (-1)^i \binom{k-1}{i-1}(i^{m-1} - i^{n-1})$$

which is the desired result. □

Applying inversion and what not we obtain:

$$k^{m-1} - k^{n-1} = \sum_{i=1}^{k} (k-1)! \binom{k-1}{i-1} [S(m,i) - S(n,i)]$$

and hence we get:

COROLLARY 6. *Let* $m, n \in \mathbf{Z}^+$, *then*

$$g(m-1, n-1) = \underset{1<k<1+\max(n,m)}{\text{g.c.d.}} \{(k-1)!(S(m,k) - S(n,k))\},$$

$$k \in \mathbf{Z}^+. \square$$

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