MULTIPLICATIVE PROPERTIES OF JENSEN MEASURES

TAKASHI ITO AND BERT M. SCHREIBER[1]

ABSTRACT. It is shown that if ψ is a linear functional on a function algebra with $\psi(1)=1$, and if ψ satisfies the Jensen inequality with respect to some finite nonnegative measure, then ψ is multiplicative.

Let A be a function algebra (closed, separating subalgebra of $C(X)$ containing the constants) on the compact space X, and let ψ be a nontrivial linear functional on A. We shall say that a finite, nonnegative measure μ on X is a *Jensen measure* for ψ if the Jensen inequality

$$|\psi(f)| \leqq \exp\left(\int_X \log|f|\, d\mu\right), \qquad f \in A, \tag{1}$$

holds for ψ and μ. It is a well-known theorem of Bishop [1], [2, §§2–5] that if ψ is multiplicative on A then there exists a Jensen representing measure μ for ψ (i.e., μ is a probability measure such that (1) holds and $\psi(f)=\int_X f d\mu, f\in A$). The purpose of this note is to point out the following converse to Bishop's theorem, thereby filling a gap in the proof of Lemma 2.5.5 in [2].

THEOREM. *If μ is a Jensen measure for ψ, then μ represents a multiplicative linear functional ϕ on A such that $\psi=\psi(1)\phi$.*

PROOF. (1) implies the continuity of ψ with $\|\psi\|\leqq 1$, so for $f\in A$ $\psi(e^{zf})$ is an entire function of the complex variable z and $\psi(e^{zf}) = \sum_0^\infty \psi(f^n)z^n/n!$. If $\int_X f d\mu=0$, then

$$|\psi(e^{zf})| \leqq \exp\left(\int_X \operatorname{Re}(zf)d\mu\right) = \exp\left(\operatorname{Re} z\int_X fd\mu\right) = 1,$$

so by Liouville's theorem we have $\psi(e^{zf})=\psi(1)$ for all z. In general, given $f\in A$ let $\alpha=\int_X f d\mu$. Then

$$\psi(e^{zf}) = \psi(\exp(z(f-\alpha\|\mu\|^{-1})))\exp(z\alpha\|\mu\|^{-1}) = \psi(1)\exp(z\alpha\|\mu\|^{-1}).$$

Thus $\sum_0^\infty \psi(f^n)z^n/n! = \psi(1)\sum_0^\infty (\alpha\|\mu\|^{-1})^n z^n/n!$, so

Received by the editors February 6, 1970.

AMS 1969 *subject classifications*. Primary 4655.

Key words and phrases. Function algebra, Jensen inequality, multiplicative linear functional.

[1] Research of the second author partially supported by NSF Grant GP-13741.

$$(2) \qquad \psi(f^n) = \psi(1)\left(\frac{1}{\|\mu\|}\int_X f d\mu\right)^n, \qquad n = 0, 1, 2, \cdots, f \in A.$$

Clearly $\psi(1) \neq 0$ if ψ is nontrivial, and from (1) we have $e^t|\psi(1)| \leq e^{t\|\mu\|}$ for all real t. Hence μ must be a probability measure, so (2) takes the form $\psi(f^n) = \psi(1)(\int_X f d\mu)^n$. Set $\phi(f) = \psi(1)^{-1}\psi(f)$, $f \in A$. Then $\phi(f) = \int_X f d\mu$ and

$$\phi(f^n) = \psi(1)^{-1}\psi(f^n) = (\phi(f))^n, \qquad n = 0, 1, 2, \cdots, f \in A.$$

It follows easily that ϕ is multiplicative on A.

Combining this result with Bishop's theorem yields

COROLLARY. *Let ψ be a linear functional on A such that $\psi(1) = 1$. Then ψ is multiplicative on A if and only if there exists a Jensen representing measure for ψ.*

REFERENCES

1. E. Bishop, *Holomorphic completions, analytic continuation, and the interpolation of semi-norms*, Ann. of Math. (2) **78** (1963), 468–500. MR **27** #4958.

2. A. Browder, *Introduction to function algebras*, Benjamin, New York, 1969. MR **39** #7431.

WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202