COUNTABLE CONNECTED SPACES

GARY GLENN MILLER

ABSTRACT. Two pathological countable topological spaces are constructed. Each is quasimetrizable and has a simple explicit quasimetric. One is a locally connected Hausdorff space and is an extension of the rationals. The other is a connected space which becomes totally disconnected upon the removal of a single point. This space satisfies the Urysohn separation property—a property between T_2 and T_3—and is an extension of the space of rational points in the plane. Both are one dimensional in the Menger-Urysohn [inductive] sense and infinite dimensional in the Lebesgue [covering] sense.

1. **Introduction.** The first example of a countable connected Hausdorff space was given by Urysohn [18]. Other examples have been given by Hewitt [7], Bing [1], Brown [2], Golomb [6], Martin [10], Roy [15], Kirch [8], Stone [17], and Miller and Pearson [13].

A connected space X has a *dispersion point* x provided $X-x$ is totally disconnected. X is a *Urysohn space* if for all distinct p and q in X there are neighborhoods U and V of p and q respectively such that U and V have disjoint closures.

The first example of a space having a dispersion point was given by Knaster and Kuratowski [9]. Such spaces have been investigated by Erdös [3] and Wilder [19]. Roy [15] has given an example of a countable connected Urysohn space having a dispersion point. Kirch [8] has given an example of a locally connected countable connected Hausdorff space. Stone [17] has also announced the construction of such an example. Recently Franklin and Krishnarao [5] have announced an interesting application of such an example due to F. B. Jones (unpublished).

The examples presented here are obtained by extending a metric space by adjoining countably many limit points and extending the distance function to a quasimetric. A real valued function $D(x, y)$ is a *quasimetric* for a topological space provided that for points x, y, z of the space:

1. $D(x, y) \geqq 0$, the equality holding iff $x=y$.
2. $D(x, y)+D(y, z) \geqq D(x, z)$.

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3. The collection of ϵ-balls, $B(x, \epsilon) = \{y : D(x, y) < \epsilon\}$, is a base for the topology of the space.

Therefore a symmetric quasimetric is a metric. There are just two essentially different ways of writing the triangle inequality (2); one implies symmetry—$D((x, y) = D(y, x)$—the other, as we have written it, does not.

The following characterization, originally due to Ribeiro [16], indicates the severity of the existence of a quasimetric. A T_1 space X is quasimetrizable iff each point p has a base for its neighborhood system $X = G_1(p), G_2(p), \cdots$ such that for any points x and y, $x \in G_{n+1}(y)$ implies $G_{n+1}(x) \subseteq G_n(y)$ for $n = 1, 2, \cdots$. Relations between Moore, metric and quasimetric spaces have been investigated by Stoltenberg [16]; related work may also be found in [4] and [14].

The following examples indicate that a quasimetric space can be quite pathological even though its distance function is the Euclidean metric when restricted to a dense open subspace.

2. **A countable locally connected quasimetric extension of the rationals.** This connected Hausdorff space has differing inductive and covering dimensions and is not a Urysohn space.

Construction. Let Q be the set of rational points on the x-axis, and M the set of points above the x-axis with both coordinates rational. Let m be a point of M. Define $F(m) = \{r, s\}$ where r and s are the points of the x-axis which together with m are the vertices of an equilateral triangle. Note that $F(m)$ does not intersect $S = Q \cup M$. Let x and y be points of S. If x is a point of Q and y a point of M, define $D(x, y) = d(x, z) + d(z, y)$ and $D(y, x) = d(z, x)$ where z is the point of $F(y)$ closest to x and d is the usual metric for the plane. Assume x and y are distinct points of M. Note that $F(x)$ and $F(y)$ are disjoint. Let z and w be points of $F(x)$ and $F(y)$ respectively such that $d(z, w) = d[F(x), F(y)]$. Define $D(x, y) = d(z, w) + d(w, y)$. Finally, if $x = y$ or x and y are points of Q, define $D(x, y) = d(x, y)$. D clearly satisfies properties (1) and (2) above. The ϵ-balls then form a base for some topology of S. Assume S has this topology. S is clearly a Hausdorff space.

LEMMA 1. *Each ϵ-ball is connected.*

PROOF. Suppose to the contrary for some p in S and positive ϵ there are nonempty disjoint open sets U and V such that $U \cup V = B(p, \epsilon)$. Assume p is a point of Q. Then $B(p, \epsilon)$ contains the interval of rationals $(p - \epsilon, p + \epsilon) \cap Q$. There are open intervals G and H on the x-axis, contained in U and V respectively, such that the usual distance from G to H is less than $\epsilon/3$. $B(p, \epsilon)$ contains all points of M with second coordinates less than $\epsilon/3$ which lie in an equilateral triangle which has base $(p-\epsilon, p+\epsilon)$. Thus there is a point q in $M \cap B(p, \epsilon)$ such that $F(q)$ intersects both G and H. Thus q is a limit point of both U and V, a contradiction.

Therefore p is in M. In this case, $B(p, \epsilon)$ consists of p together with two sets each like the ϵ-ball in the above. By a similar argument, each of the two sets is connected. Furthermore p is a limit point of the two sets. This involves a contradiction.

COROLLARY. *S is a countable connected Hausdorff space which is locally connected, quasimetric and contains Q as a dense subspace.*

3. A countable connected quasimetric extension of $Q \times Q$ having a dispersion point. This space is Urysohn and also has differing dimensions.

Construction. Let $X = (Q \times Q) \cup Z \cup \{\omega\}$ where Q denotes the rationals, Z the integers and $\omega = (\pi, \sqrt{2})$. X endowed with the topology described below is the desired example. Let F be a function from Z into the plane such that (1) for each z in Z, $F(z)$ is not in X and the only image of F on the line $\omega F(z)$ is $F(z)$, and (2) $F[Z]$ is dense in the plane. For each z in Z, let $G(z)$ be the midpoint of the line segment $\omega F(z)$. A quasimetric D for X is defined next in terms of the usual metric d for the plane. First let d^* be the usual bounded metric for the plane, $d^* = d/1+d$. Let x and y be points of X, let a and b be points of $X-Z$ and let z be a point of Z. Define D as follows.

1. $D(a, b) = d^*(a, b)$.
2. $D(z, a) = \min[d^*(a, F(z)), d^*(a, G(z))]$.
3. $D(x, y) = 1$ for $x \neq y$.
4. $D(x, x) = 0$.

The triangle inequality and positive definite property follow immediately. Let X have the topology induced by the quasimetric D.

Notice if ϵ is a positive number less than one, an ϵ-ball with center a point of $M = (Q \times Q) \cup \{\omega\}$ is just the common part of M and an open disc with center the point. An ϵ-ball with center a point z of Z is just the union of two such rational discs with centers $F(z)$ and $G(z)$ together with z. Thus $Q \times Q$ is dense in X, Z is closed in X, and each have their usual topologies as subspaces of X.

LEMMA 2. *X is connected.*

PROOF. Suppose X is the union of two nonempty disjoint open sets U and V such that U contains ω. Since each point of Z is a limit point of $Q \times Q$, there is a point x_0 in $Q \times Q \cap V$. For each point x in the plane, let $S(x, \epsilon)$ denote the ϵ-disc with center x relative to the usual metric d. There exist a point z_1 of Z and a positive ϵ_1 such that $S(G(z_1), \epsilon_1) \cap Q \times Q$ lies in V and $\epsilon_1 < t/2$ where $t = d(\omega, F(z_1))$. Let x_1 be a common point of $S(G(z_1), t_1)$ and $Q \times Q$. There exist a point z_2 of Z and a positive $\epsilon_2 < t/2^2$ such that $F(z_2)$ is in $S(G(z_1), \epsilon_1)$ and $S(G(z_2), \epsilon_2) \cap Q \times Q$ lies in V. Let x_2 be a common point of $S(G(z_2), \epsilon_2)$ and $Q \times Q$. Continue this process. It is easily seen that for each natural number n, $d(\omega, x_n) < [n+1]t/2^n$. Therefore $x_1, x_2, \cdots$ converges to ω in the plane. Therefore ω is a limit point of V in X, which is a contradiction.

LEMMA 3. *If x and y are points of $X - \{\omega\}$, then $X - \{\omega\}$ is the union of disjoint open sets U and V containing x and y respectively.*

PROOF. Since $\omega = (\sqrt{2}, \pi)$, and π is transcendental, a line through ω containing a point of $Q \times Q$ cannot have rational slope. Therefore each line through ω contains at most one point of $Q \times Q$. Let L be a line through ω such that L does not intersect $(Q \times Q) \cup F[Z]$ and (1) L separates x and y if both are in $Q \times Q$, (2) L separates $F(x)$ and $F(y)$ if x and y are in Z, and (3) L separates $F(x)$ and y if x is in Z and y is in $Q \times Q$. Let M_1 be the set of all points of $Q \times Q$ on one side of L and M_2 the set of all such points on the other. Let N_i be the set of all z in Z such that $F(z)$ is on the M_i side of L for $i = 1, 2$. Let $U = M_1 \cup N_1$ and $V = M_2 \cup N_2$. Then U and V are disjoint and open in $X - \omega$, $X - \omega = U \cup V$, and x is in one of U and V and y is in the other.

LEMMA 4. *X is a Urysohn space.*

PROOF. Suppose x is a point of $X - Z$. Let U and V be the intersections of $X - Z$ with open discs in the plane having centers x and ω and each having radius $\epsilon = d(\omega, x)/4$. Clearly, no point of $X - Z$ is a limit point in X of both U and V. Suppose some point z in Z is a limit point of U. Then $d(x, F(z)) \leqq \epsilon$ or $d(x, G(z)) \leqq \epsilon$. In either case $d(\omega, F(z)) > \epsilon$ and $d(\omega, G(z)) > \epsilon$. Thus z is not a limit point of V. Therefore $\overline{U} \cap \overline{V} = \varnothing$.

Now suppose x is a point of Z. Let U_1, U_2 and V be the intersections of $X - Z$ with open discs in the plane having centers $F(x)$, $G(x)$ and ω respectively and each having radius $d(G(x), \omega)/4$. From the first part of the proof, $\overline{U}_1 \cap \overline{V} = \overline{U}_2 \cap \overline{V} = \varnothing$. Let $U = U_1 \cup U_2 \cup \{x\}$. Then $\overline{U} \cap \overline{V} = (\overline{U}_1 \cap \overline{V}) \cup (\overline{U}_2 \cap \overline{V}) \cup (\{x\} \cap \overline{V}) = \varnothing$.

Finally, suppose x and y are points of $X - \omega$. From Lemma 3, there are disjoint open sets U_1 and V containing x and y respectively such that $X - \omega = U_1 \cup V$. There are disjoint open sets U_2 and V_2 containing x and ω respectively. Let $U = U_1 \cap U_2$. Then $\overline{U} \cap \overline{V} = \varnothing$.

COROLLARY. *X is a countable connected quasimetric Urysohn space which has a dispersion point and contains $Q \times Q$ as a dense subspace.*

4. Dimension.

LEMMA 5. *S and X are each one dimensional in the Menger-Urysohn sense.*

PROOF. In each space, each point has arbitrarily small ϵ-neighborhoods with boundaries consisting of isolated points in the relative topology.

LEMMA 6. *S and X are each infinite dimensional in the Lebesgue sense.*

PROOF. Let P be the set of all points in S with second coordinate greater than 1. Then P is closed and homeomorphic to Z. Let n be a positive integer. Then P can be partitioned into subsets $P_1, \cdots,$ P_n such that for each relatively open set U in Q, P_i contains a limit point of U for $i=1, \cdots, n$. Let $U_i=(S-P)\cup P_i$ for $i=1, \cdots, n$. Then $U_1, \cdots, U_n$ is an open cover of S. Suppose $\mathcal{G}$ is an open cover of S which refines this cover. Let G_1 be a member of $\mathcal{G}$ which lies in U_1. Then $Q\cap G_1$ is a nonempty open set since Q is open and dense. Let p_2 be a limit point of $Q\cap G_1$ in P_2. Let G_2 be a member of $\mathcal{G}$ which contains p_2. Then G_2 lies in U_2. $Q\cap G_1\cap G_2$ is nonempty since p_2 is a limit point of $Q\cap G_1$. Thus $Q\cap G_1\cap G_2$ has a limit point p_3 in P_3. Continuing in this manner, we have distinct members $G_1, \cdots, G_n$ of $\mathcal{G}$ with a common point. Thus the order of $\mathcal{G}$ is at least n. Since this holds for each positive integer, S is infinite dimensional in the Lebesgue sense.

The preceding argument when modified by replacing P by Z, Q by $Q \times Q$ and S by X establishes the desired result for X.

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University of Victoria, Victoria, British Columbia, Canada