OSCILLATION CRITERIA FOR NONLINEAR MATRIX DIFFERENTIAL EQUATIONS

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Abstract. Oscillation criteria are established for nonlinear matrix differential equations of the form $[R(t)U']' + F(t, U, U') = 0$. These criteria are more general than some similar ones of E. C. Tomastik insofar as they do not require $F$ to be positive definite.

In [1] Tomastik derives oscillation criteria for nonlinear matrix differential equations of the form

$$[R(t)U']' + F(t, U, U')U = 0$$

under the hypothesis that the matrix $F$ is positive definite. The purpose of this note is to present an oscillation criterion for (1) which does not require such an assumption.

As in [1] we consider a “prepared” solution $U(t)$ of (1) satisfying

$$U^*(t)R(t)U'(t) = U^*(t)R(t)U(t),$$

and say that (1) is oscillatory on $[a, \infty)$ if the determinant of every prepared solution has arbitrarily large zeros. Our assumptions regarding the coefficient matrices are as in [1]. In particular, $R$ and $F$ are to be sufficiently regular, symmetric and real $n \times n$ matrices, and $R(t)$ is to be positive definite for all $t$.

The oscillation criterion to be derived below depends on a comparison of solutions of (1) and an equation of the same type,

$$[P(t)V']' + G(t, V, V')V = 0.$$  

Lemma 1. Let $U(t)$ be a prepared matrix solution of (1) such that

$$\det U(t) \neq 0$$

on some interval $[b, c]$, and let $S(t) = R(t)U'(t)U^{-1}(t)$. If $V(t)$ satisfies (3) on $[b, c]$, then

$$[V^*P' - V^*S]_{t=b}^{t=c} = \int_{b}^{c} V^*(F - G)V' dt + \int_{b}^{c} V^*(P - R)V' dt$$

$$+ \int_{b}^{c} (V' - U'U^{-1}V)^*R(V' - U'U^{-1}V) dt.$$  

Proof. If $\det U(t) \neq 0$, then a direct computation using (2) and the
fact that \( U^{-1} = -U^{-1}U'U^{-1} \) yields the following Picone-type identity
\[
(V^*PV' - V^*RU'U^{-1}V)' = V^*(PV')' - V^*(RU')'U^{-1}V + V^*(P - R)V' + (V' - U'U^{-1}V)^*R(V' - U'U^{-1}V).
\]

Substituting (1) and (3) for the first two terms on the right side of this equation and integrating, (4) follows readily.

**Lemma 2.** Suppose \( V(t) \) is a nontrivial solution of (3) satisfying

(i) \( V^*(t) [F(t, U, U') - G(t, U, U')] V(t) \) is positive semidefinite for \( b \leq t \leq c \) and all values of \( U \) and \( U' \),

(ii) \( V^*(t) [P(t) - R(t)] V(t) \) is positive semidefinite for \( b \leq t \leq c \),

(iii) \( F(b) = F(c) = 0 \).

If \( U(t) \) is a prepared solution of (1), then \( \det U(t) \) has a zero in \( [b, c] \).

**Proof.** If \( \det U(t) \neq 0 \) in \( [b, c] \), then (4) holds and our hypotheses assure that the left side of (4) is 0 while each term on the right side of (4) is positive semidefinite. Furthermore, the last term on the right side of (4) is zero if and only if \( V' - U'U^{-1}V = 0 \) on \( [b, c] \), and this requires \( V'(b) = 0 \). By the uniqueness theorem for matrix systems, \( V(b) = V'(b) = 0 \) implies \( V(t) = 0 \), contradicting the hypotheses and showing that \( \det U(t) = 0 \) for some \( t \) in \( [b, c] \).

Our oscillation criteria for (1) now follow easily by comparing (1) with the linear matrix equation
\[
(3') \quad [p(t)IV']' + g(t)IV = 0.
\]

Let \( J \) be a nonzero matrix with zeros and ones down the main diagonal and zeros elsewhere.

**Theorem.** If the Sturm-Liouville equation \( (p(t)v')' + g(t)v = 0 \) is oscillatory at \( t = \infty \), and if for some real \( a \) and some \( J \)

(i) \( J [F(t, U, U') - g(t)I] J \) is positive semidefinite for \( t \geq a \) and all values of \( U \) and \( U' \),

(ii) \( J [p(t)I - R(t)] J \) is positive semidefinite for \( t \geq a \), then (1) is oscillatory on \( [a, \infty) \).

**Proof.** Let \( v(t) \) be a nontrivial solution of \( (p(t)v')' + g(t)v = 0 \) which is oscillatory at \( \infty \) and define \( V(t) = v(t)J \). Then \( V(t) \) satisfies \( (3') \), and we can find arbitrarily large pairs of numbers \( (b, c) \) satisfying \( c > b > a \) for which \( V(b) = V(c) = 0 \). Furthermore
\[
V^*[F - gI]V = v^*J[F - gI]J \quad \text{and} \quad V^*[pI - R]V' = v'^*J[pI - R]J
\]
so that conditions (i) and (ii) of Lemma 2 are satisfied on \( [a, \infty) \). By Lemma 2, equation (1) is oscillatory on \( [a, \infty) \).
In order to apply this Theorem, it is useful to recall the Leighton oscillation criterion: if

$$\int^{\infty}_{0} \frac{1}{p(t)} \, dt = \int^{\infty}_{0} g_1(t) \, dt = \infty,$$

then \((p_1v)' + g_1v = 0\) is oscillatory at \(t = \infty\). Consider now the system (1) where \(n = 2\) and

$$0 < R(t) \equiv \begin{pmatrix} p_1(t) & 0 \\ 0 & p_2(t) \end{pmatrix}, \quad F(t, U, U') \equiv \begin{pmatrix} g_1(t) & 0 \\ 0 & g_2(t) \end{pmatrix}.$$          

According to the Theorem with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

if (5) is satisfied then (1) is oscillatory. This result does not follow from [1].

REFERENCES


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