ON NONINVERTIBLE LINKS WITH INVERTIBLE PROPER SUBLINKS

W. C. WHITTEN, JR.

ABSTRACT. The title supports the primary objective of this paper. Specifically, for each $\mu \geqq 3$, a noninvertible, oriented, ordered link of μ components tamely imbedded in S^3 is exhibited with the property that each proper sublink is invertible. The case $\mu = 2$ has been covered in a previous paper.

In 1963, H. F. Trotter gave a solution to a difficult problem of classical knot theory by exhibiting a collection of noninvertible knots [8]. More recently examples of noninvertible links were given [9], each link consisting of two components each of which is invertible. An oriented, ordered link L of μ components in the oriented 3-sphere S is *invertible* if there is an orientation-preserving autohomeomorphism of S taking each component of L onto itself with reversal of orientation. In this paper we solve a "Brunnian type" problem on invertibility of links by presenting for each $\mu \geqq 3$ a noninvertible link of μ components with the feature that each proper sublink is invertible. Such a link is shown in Figure 1.

I wish to thank W. R. Alford and K. Murasugi for conversations. I would like also to thank the referee for a helpful comment concerning notation.

REMARK. The link L of Figure 1 is assumed to have $\mu \geqq 3$ components. Note that each component is of trivial knot type in S, and that each proper sublink of L is invertible. We now proceed to prove that L itself, however, is noninvertible.

Let V denote a closed solid torus tamely imbedded in $S - K_\mu$. The core of V is to be of knot type 5_1. If V' is a tubular neighborhood of the component K_μ' of the link L' of Figure 2 and if the closure of V' does not otherwise meet L', then we make the further requirement of V that there be a homeomorphism of the closure of the complement of V' in S onto V which takes K_α' onto K_α $(\alpha = 1, \cdots, \mu - 1)$. A section of V is indicated by the heavy black lines in the lower right-hand portion of Figure 1.

Now let λ denote the core of V and set $\Lambda = K_\mu \cup \lambda$. Orient λ as shown in Figure 3. We have

Received by the editors January 6, 1970.

AMS 1970 *subject classifications*. Primary 55A25; Secondary 57A10.

Key words and phrases. Classical knot theory, noninvertible knots, noninvertible links, "Brunnian type" invertibility problem.

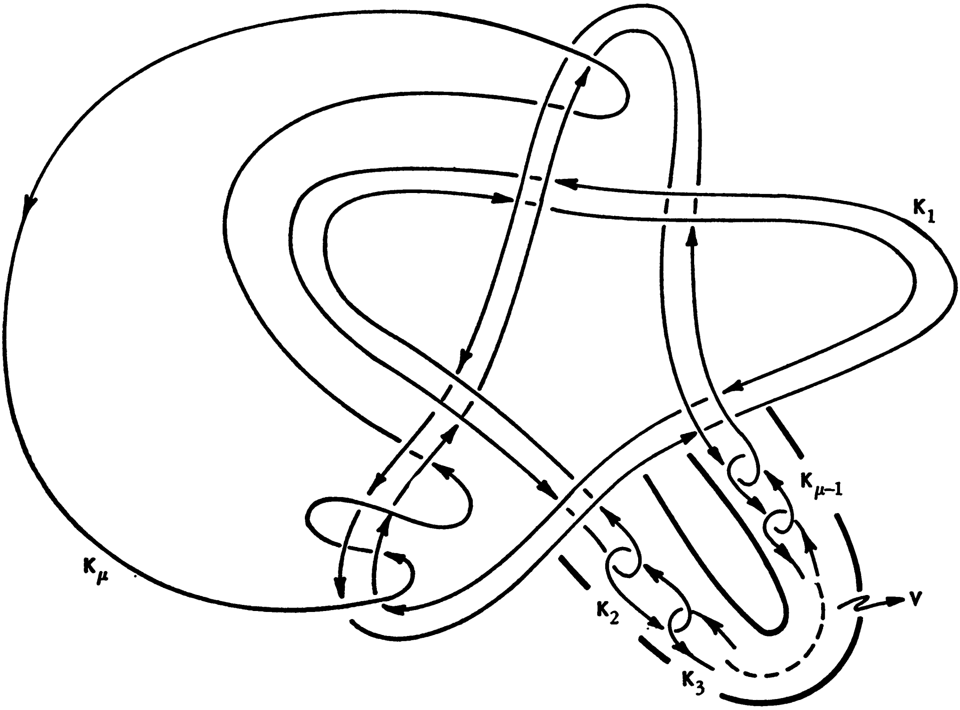

FIGURE 1

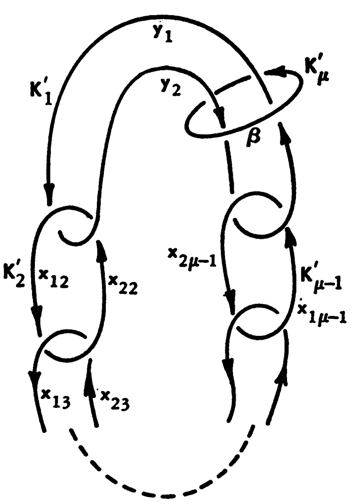

FIGURE 2

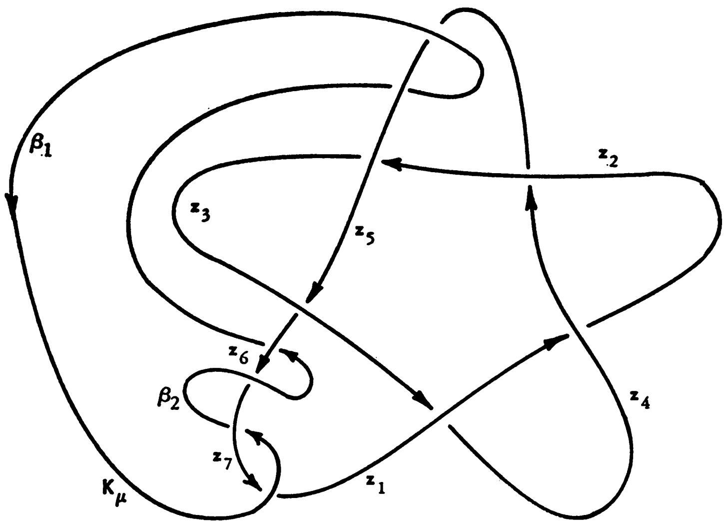

FIGURE 3

$$\begin{aligned}\pi_1(S-(K_\mu\cup \operatorname{Int} V)) &\approx \pi_1(S-\Lambda)\\ &= \big|\,\beta_1,\beta_2,z_1,\cdots,z_7\colon \beta_1 z_7\beta_2^{-1}z_7^{-1}, \beta_2 z_6 z_5^{-1}\beta_1^{-1}z_5 z_6^{-1},\\ &\qquad z_1z_4z_2^{-1}z_4^{-1},\ z_2z_5z_3^{-1}z_5^{-1}, z_3z_1z_4^{-1}z_1^{-1},\\ &\qquad z_4z_2\beta_1^{-1}z_5^{-1}\beta_1 z_2^{-1},\ z_5z_3z_6^{-1}z_3^{-1}, z_6\beta_2z_7^{-1}\beta_2^{-1},\ z_7\beta_1z_1^{-1}\beta_1^{-1}\,\big|.\end{aligned} \tag{1}$$

The group $\pi_1(V-L^*)$, where $L^*=K_1\cup\cdots\cup K_{\mu-1}$, is isomorphic to the group of the link in Figure 2, so that

$$\begin{aligned}\pi_1(V-L^*) \approx \big|\,&\beta, x_{12}, x_{22},\cdots, x_{1\mu-1}, x_{2\mu-1}, y_1, y_2\colon y_1x_{12}^{-1}y_2^{-1}x_{12},\\ &y_2x_{22}y_2^{-1}x_{12}^{-1},\ x_{12}x_{13}^{-1}x_{22}^{-1}x_{13},\ x_{22}x_{23}x_{22}^{-1}x_{13}^{-1},\cdots,\\ &x_{1\mu-2}x_{1\mu-1}^{-1}x_{2\mu-2}^{-1}x_{1\mu-1},\ x_{2\mu-2}x_{2\mu-1}x_{2\mu-2}^{-1}x_{1\mu-1}^{-1},\\ &x_{1\mu-1}\beta^{-1}y_1^{-1}\beta x_{2\mu-1}^{-1}\beta^{-1}y_1\beta,\ x_{2\mu-1}\beta^{-1}y_2\beta x_{2\mu-1}^{-1}\beta^{-1}y_1^{-1}\beta,\\ &\beta(y_1^{-1}y_2)\beta^{-1}(y_1^{-1}y_2)^{-1}\,\big|.\end{aligned} \tag{2}$$

For the link $K_1'\cup\cdots\cup K_{\mu-1}'$, which is a proper sublink of that in Figure 2, we have

$$\begin{aligned}\pi_1(S-K_1'\cup\cdots\cup K_{\mu-1}') = \big|\,&y_1,y_2,x_{12},x_{22},\cdots,x_{1\mu-1},\\ &x_{2\mu-1}\colon y_1x_{12}^{-1}y_2^{-1}x_{12},\ y_2x_{22}y_2^{-1}x_{12}^{-1},\ x_{12}x_{13}^{-1}x_{22}^{-1}x_{13},\\ &x_{22}x_{23}x_{22}^{-1}x_{13}^{-1},\ x_{13}x_{14}^{-1}x_{23}^{-1}x_{14},\ x_{23}x_{24}x_{23}^{-1}x_{14}^{-1},\cdots,\\ &x_{1\mu-2}x_{1\mu-1}^{-1}x_{2\mu-2}^{-1}x_{1\mu-1},\ x_{2\mu-2}x_{2\mu-1}x_{2\mu-2}^{-1}x_{1\mu-1}^{-1},\\ &x_{1\mu-1}y_1^{-1}x_{2\mu-1}^{-1}y_1,\ x_{2\mu-1}y_2x_{2\mu-1}^{-1}y_1^{-1}\,\big|.\end{aligned} \tag{3}$$

In order to prove that $\pi_1(S-L)$ is the free product of a certain pair of groups with amalgamated subgroup (see the theorem below), we need the intuitively obvious fact contained in the following

LEMMA. *The element $y_1^{-1}y_2$ of the group $\pi_1(S-K_1'\cup\cdots\cup K_{\mu-1}')$ of* (3) *is not the identity.*

PROOF. For convenience let $\pi_1(S-K_1'\cup\cdots\cup K_{\mu-1}')$ be denoted by D_μ. Then

$$\begin{aligned}D_3 &= \big|\,Y_1,Y_2,X_{12},X_{22}\colon Y_1X_{12}^{-1}Y_2^{-1}X_{12},\ Y_2X_{22}Y_2^{-1}X_{12}^{-1},\\ &\qquad X_{12}Y_1^{-1}X_{22}^{-1}Y_1,\ X_{22}Y_2X_{22}^{-1}Y_1^{-1}\,\big|\\ &= \big|\,Y_1,X_{12}\colon X_{12}Y_1^{-1}X_{12}Y_1^{-1} = Y_1^{-1}X_{12}Y_1^{-1}X_{12}\,\big|.\end{aligned}$$

Since it is assumed that $\mu \geqq 3$ always, there is a rather natural homomorphism of $\pi_1(S-K_1' \cup \cdots \cup K_{\mu-1}')$ onto D_3 given by

$$y_1, x_{13}, x_{15}, \cdots, x_{1\mu-2} \to Y_1$$

$$y_2, x_{23}, x_{25}, \cdots, x_{2\mu-2} \to Y_2$$

$$x_{12}, x_{14}, x_{16}, \cdots, x_{1\mu-1} \to X_{12}$$

$$x_{22}, x_{24}, x_{26}, \cdots, x_{2\mu-1} \to X_{22},$$

provided μ is odd. Thus, if μ is odd and $y_1^{-1}y_2 = 1$ in

$$\pi_1(S - K_1' \cup \cdots \cup K_{\mu-1}'),$$

then $D_3 \approx D_3/\langle Y_1^{-1}Y_2\rangle$. ($\langle Y_1^{-1}Y_2\rangle$ is the consequence in D_3 of $Y_1^{-1}Y_2$.) But the Alexander polynomial of D_3 is $X-Y$ and that of $D_3/\langle Y_1^{-1}Y_2\rangle$ is 1; (this latter group is free abelian of rank two).

Now if μ is even then, similar to above, we have a homomorphism of D_μ onto D_4, or D_6, or D_8 according as $\mu-1$ is of the form $3+6n$, or $5+6n$, or $7+6n$, respectively, where n is a nonnegative integer. To conclude the proof of the lemma, we need only show that $D_k \not\approx D_k/\langle Y_1^{-1}Y_2\rangle$, ($k=4, 6, 8$). But straightforward and easy calculations reveal that the Alexander ideal (see p. 209 of [2]) of $D_k/\langle Y_1^{-1}Y_2\rangle$ for each of $k=4, 6, 8$ is not principal. This concludes the proof of the lemma.

THEOREM. *The group $\pi_1(S-L)$ of the link $L=K_1 \cup \cdots \cup K_\mu$ of Figure 1 is isomorphic to the free product of the groups $\pi_1(S-(K_\mu \cup \mathrm{Int}\ V))$ and $\pi_1(V-L^*)$ amalgamated with respect to $\pi_1(\partial V)$, where L^* denotes the proper sublink $K_1 \cup \cdots \cup K_{\mu-1}$ of L.*

PROOF. Choose a point x on ∂V as common basepoint for the four groups. Taking x also as the basepoint of the group $\pi_1(S-\mathrm{Int}\ V)$, it follows from [6] and [1] that the inclusion homomorphism $\pi_1(\partial V) \to \pi_1(S-\mathrm{Int}\ V)$ is a monomorphism, since V is knotted in S; hence, *a fortiori*, the inclusion homomorphism $\pi_1(\partial V) \to \pi_1(S-(K_\mu \cup \mathrm{Int}\ V))$ is a monomorphism.

From (2) and the definition of V it is clear that an element of $\pi_1(\partial V)$ can be written in the form $\beta^r(y_1^{-1}y_2)^s$, where r and s are integers. Killing β in (2) maps $\pi_1(V-L^*)$ homomorphically onto the group $\pi_1(S-K_1' \cup \cdots \cup K_{\mu-1}')$, (see Figure 2), and sends $\beta^r(y_1^{-1}y_2)^s$ onto $(y^{-1}y_2)^s$. If $\beta^r(y_1^{-1}y_2)^s = 1$ in $\pi_1(V-L^*)$, then $(y_1^{-1}y_2)^s = 1$ in $\pi_1(S-K_1' \cup \cdots \cup K_{\mu-1}')$. Since, by the lemma, $y_1^{-1}y_2 \neq 1$ in this latter group, it follows from Corollary (31.9), p. 23, of [6] that $s=0$. Hence, $\beta^r(y_1^{-1}y_2)^s = 1$ in $\pi_1(V-L^*)$ implies that $\beta^r = 1$. Since $\beta \neq 1$ in $\pi_1(V-L^*)$ and since $\pi_1(V-L^*)$ is the group of a link in S, it follows, again from

[6], that $r=0$. Thus, the inclusion $\partial V \rightarrow V-L^*$ induces a monomorphism $\pi_1(\partial V) \rightarrow \pi_1(V-L^*)$.

The theorem follows by an application of van Kampen's theorem [5].

From the theorem it follows immediately that the inclusion map $S-(K_\mu \cup \operatorname{Int} V) \rightarrow S-L$ induces a monomorphic imbedding of $\pi_1(S-(K_\mu \cup \operatorname{Int} V))$ in $\pi_1(S-L)$. As common basepoint for these two groups choose a point x on ∂V.

We consider the covering space Σ of $S-(K_1 \cup \cdots \cup K_{\mu-1})$ branched over K_μ whose associated unbranched covering space of $S-L$ arises from the conjugacy class of subgroups of $\pi_1(S-L)$ to which $\pi_1(S-(K_\mu \cup \operatorname{Int} V))$ belongs (see [3] or §8 of [4]). If Ξ denotes the set of points in Σ lying over K_μ, then (1) gives a presentation for $\pi_1(\Sigma-\Xi)$. From the geometric interpretation of the generators of this presentation (see Figures 1 and 3), it is clear that a presentation for $\pi_1(\Sigma)$ is obtained from (1) simply by killing β_1 and β_2. Consequently,

$$\pi_1(\Sigma) \approx \left| z_1, z_4 : z_1 z_4 z_1 z_4 z_1 = z_4 z_1 z_4 z_1 z_4 \right| . \tag{4}$$

Now the element $w = z_7 z_6 z_5^{-1}$ in $\pi_1(S-(K_\mu \cup \operatorname{Int} V)) \subset \pi_1(S-L)$ is clearly a longitude of the component $+K_\mu$ of L (see Figures 1 and 3). Thus, any orientation-preserving autohomeomorphism ψ of S inverting L induces an automorphism of $\pi_1(S-L)$ taking w onto a conjugate of its inverse. In fact we can so choose a path from the basepoint x to its image $\psi(x)$ that the automorphism induced by ψ takes w onto w^{-1}. This means then that the automorphism of $\pi_1(\Sigma)$, induced by the autohomeomorphism of Σ (determined up to a covering translation) which in turn is induced by ψ, can be assumed to carry the element

$$v = z_1^2 (z_1 z_4) z_1^{-1} (z_1 z_4)^{-1}$$

of $\pi_1(\Sigma)$ onto its inverse. v is the element of $\pi_1(\Sigma)$ represented by a path in $\Sigma-\Xi$ representing w in $\pi_1(\Sigma-\Xi)$. The groups $\pi_1(\Sigma-\Xi)$ and $\pi_1(\Sigma)$ are assumed to have the same basepoint in $\Sigma-\Xi$, this common basepoint lying over x in ∂V.

Under the assumption then that L is invertible, it follows that

(5) *There is an automorphism ϕ of $\pi_1(\Sigma)$ with the property that $\phi(v) = v^{-1}$.*

We shall prove that (5) is not true.

Referring to (4), it is seen that

$$\pi_1(\Sigma) \approx \left| \sigma, \omega : \sigma^2 = \omega^5 \right| ,$$

where $z_1 = \omega^{-2}\sigma$, $z_4 = \sigma^{-1}\omega^3$. By the work of Schreier [7] each automorphism of this group can be written in the form

$$\sigma \longrightarrow \gamma\sigma^{\epsilon}\gamma^{-1}, \qquad \omega \longrightarrow \gamma\omega^{\epsilon}\gamma^{-1} \qquad (\epsilon = \pm 1),$$

for some γ in $\pi_1(\Sigma)$. Writing v in terms of σ and ω we obtain

$$v = (\omega^{-2}\sigma)^3\sigma^{-1}\omega^3\sigma^{-1}\omega,$$

and it follows from (5) and what has been said that

(6) *There is an element γ in $\pi_1(\Sigma)$ such that either*
(a) $\gamma[(\omega^{-2}\sigma)^3\sigma^{-1}\omega^3\sigma^{-1}\omega]\gamma^{-1} = \omega^{-1}\sigma\omega^{-3}\sigma(\sigma^{-1}\omega^2)^3$, *or*
(b) $\gamma[(\omega^2\sigma^{-1})^3\sigma\omega^{-3}\sigma\omega^{-1}]\gamma^{-1} = \omega^{-1}\sigma\omega^{-3}\sigma(\sigma^{-1}\omega^2)^3$.

It was shown, however, in [9] that (6b) cannot hold. A similar argument shows that (6a) also cannot hold, and we conclude that the link L of Figure 1 is noninvertible.

References

1. M. Dehn, *Über die Topologie des drei-dimensionalen Raumes*, Math. Ann. **69** (1910), 137–168.
2. R. H. Fox, *Free differential calculus*. II: *The isomorphism problem of groups*, Ann. of Math. (2) **59** (1954), 196–210. MR **15**, 931.
3. ———, *Covering spaces with singularities*, Algebraic geometry and topology, A symposium in honor of S. Lefschetz, Princeton Univ. Press, Princeton, N.J., 1957, pp. 243–257. MR **23** #A626.
4. ———, *A quick trip through knot theory*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 120–167. MR **25** #3522.
5. E. R. van Kampen, *On the connection between the fundamental groups of some related spaces*, Amer. J. Math. **55** (1933), 261–267.
6. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **66** (1957), 1–26. MR **19**, 761.
7. O. Schreier, *Über die Gruppen $A^aB^b = 1$*, Abh. Math. Sem. Univ. Hamburg **3** (1924), 167–169.
8. H. F. Trotter, *Non-invertible knots exist*, Topology **2** (1964), 275–280. MR **28** #1618; errata **30**, p. 1203.
9. W. C. Whitten, Jr., *A pair of non-invertible links*, Duke Math. J. **36** (1969), 695–698.

University of Southwestern Louisiana, Lafayette, Louisiana 70501