OPERATORS FROM BANACH SPACES TO COMPLEX INTERPOLATION SPACES

VERNON WILLIAMS

Abstract. Given a closed linear operator $A$ with dense domain in a Banach space $X$, M. Schechter [4] utilized the Lebesgue integral to construct a family of bounded linear operators from $X$ to the Calderón complex interpolation space $(X, D(A))_* [2]$, where $D(A)$, the domain of $A$ in $X$, is a Banach space under the norm

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

In this paper we utilize the complex functional calculus, which provides a more natural setting, to construct a similar family of operators. At the same time we achieve a strengthening of the Schechter result, for in the proof of our theorem we make no use of the adjoint $A^*$ of $A$ and consequently do not require the domain of $A$ to be dense in $X$. A completely analogous procedure would permit the removal from the Schechter theorem, referred to above, of the hypothesis that the domain of $A$ is dense in $X$.

Given a closed linear operator $A$ with dense domain in a Banach space $X$, M. Schechter [4] utilized the Lebesgue integral to construct a family of bounded linear operators from $X$ to the Calderón complex interpolation space $(X, D(A))_* [2]$, where $D(A)$, the domain of $A$ in $X$, is a Banach space under the norm

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

In this paper we utilize the complex functional calculus, which provides a more natural setting, to construct a similar family of operators. At the same time we achieve a strengthening of the Schechter result, for in the proof of our theorem we make no use of the adjoint $A^*$ of $A$ and consequently do not require the domain of $A$ to be dense in $X$. A completely analogous procedure would permit the removal from the Schechter theorem referred to above of the hypothesis that the domain of $A$ is dense in $X$.

Theorem. (a) Let $X$ be a complex Banach space. Let $A$ be a closed linear operator in $X$ such that the resolvent set of $A$ is nonempty, and the spectrum of $A$ is contained in a Cauchy domain; that is, a subset $D$ of the complex plane with the following properties:

Received by the editors October 18, 1969.

AMS 1969 subject classifications. Primary 4725, 4638; Secondary 4730.

Keywords and phrases. Calderón complex interpolation space, functional calculus, closed linear operator with spectrum contained in a Cauchy domain.

248
(1) $D$ is open.
(2) $D$ has a finite number of components, the closures of any two of which are disjoint.
(3) The boundary $B(D)$ of $D$ is composed of a finite number of closed rectifiable Jordan curves, no two of which intersect.
(b) Let $\phi(w, z)$ be a complex-valued function defined for each $w$ in the closure of the open strip $S$ in the complex plane between 0 and 1, and for each $z$ in a set $N$ containing a neighborhood of the closure of the Cauchy domain $D$, together with a neighborhood of infinity. Suppose further that $\phi(w, z)$ satisfies the following:
(1) There exists a constant $C$ such that for $w_1, w_2$ in the closure of the open strip $S$, and $z$ in $B(D)$, then
$$|\phi(w_1, z) - \phi(w_2, z)| \leq C|w_1 - w_2|.$$
(2) Let $w$ in the closure of $S$ be fixed, then $\phi(w, z)$ is an analytic function of $z$ in a neighborhood of the closure of $D$ and at infinity.
(3) For each $w$ in $S$, and each $z$ in $B(D)$, the partial derivative $\phi_w(w, z)$ exists and is continuous in $z$ on $B(D)$. We assume further that
$$\lim_{h \to 0} \frac{|\phi(w + h, z) - \phi(w, z)|}{h} - \phi_w(w, z)$$
converges to 0 uniformly in $z$ on $B(D)$ as $h$ approaches 0.
(4) For $j = 0, 1$, there exist constants $K_j$ such that for $t$ real and $z$ in $B(D)$,
$$|\phi(j + it, z)| \leq K_j.$$
(c) $B(D)$ is rectifiable by assumption (a). Let $L$ denote the length of $B(D)$. Let $M$ denote the maximum value achieved by the continuous function $\|(z - A)^{-1}\|$ on the compact set $B(D)$. Let $s$ satisfying $0 < s < 1$ be fixed. $B(D)$ is compact. Let $T$ denote the maximum of $\{|z| \mid z \text{ is a member of } B(D)\}$.
Then $\phi(w, z)$ induces a bounded linear mapping $\Phi_1(A)$ from $X$ to the Calderón complex interpolation space $(X, D(A))_s$ with norm
$$\leq \max \{K_0 ML, K_1 L(1 + M + TM)\}.$$
Remark. Observe that in particular an arbitrary constant function $\phi(w, z)$ satisfies the four conditions in assumption (b). We note further that $\phi(w, z)$ satisfies all conditions in assumption (b) if $\phi(w, z) = f(w)g(z)$ where $f(w)$ is such that:
(1) $f(w)$ is analytic in the open strip $S$.
(2) $f(w)$ satisfies a Lipschitz condition in the closure of $S$.
(3) There exist constants $K_j$, $j = 0, 1$, such that $|f(j + it)| \leq K_j$ for all real $t$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(For example $f(w) = 1/(w+1)^n$, $n$ a positive integer, and $g(z)$ is analytic in a neighborhood of the closure of $D$, and at infinity.)

(For example $g(z) = 1/(z-a)^n$, where $n$ is a positive integer and "a" is any point of the resolvent set of $A$ which does not belong to the closure of the Cauchy domain $D$.)

Proof of Theorem. We first recall the definition of the Calderón interpolation space $(X, D(A))_s$ [1].

(1) $X + D(A) = \{x_0 + x_1 | x_0 \in X$ and $x_1 \in D(A)\}$ is a Banach space under the norm

$$\|x\|_{X + D(A)} = \inf_{x_0 + x_1 = x} \{\|x_0\|_X + \|x_1\|_{D(A)}\}.$$

(2) Let $H(X, D(A))$ be the set of all functions $f$ from the closure of the open strip $S$ to $X + D(A)$ which satisfy:

(i) $f$ is analytic in $S$.

(ii) $f$ is continuous on the closure of $S$.

(iii) For each real $t$, $f(it) \in X$, $f(1+it) \in D(A)$ and there exist constants $C_0, C_1$ such that for all real $t$ $\|f(it)\|_X \leq C_0$ and $\|f(1+it)\|_{D(A)} \leq C_1$.

$H(X, D(A))$ is a Banach space under the norm

$$\|f\|_{H(X, D(A))} = \max \left\{ \sup_{t} \|f(it)\|_X, \sup_{t} \|f(1+it)\|_{D(A)} \right\}.$$

(3) $(X, D(A))_s = \{x | x = f(s), f \in H(X, D(A))\}$ is a Banach space under the norm $\|x\|_{(X, D(A))_s} = \inf_{f(s) \rightarrow x} \{\|f\|_{H(X, D(A))}\}$.

We show in the sequel that each of the following is valid:

(I) For $w$ fixed in the closure of $S$,

$$\phi(w, A) = \int_{B(D)} \phi(w, z)(z - A)^{-1}dz$$

is a bounded linear operator mapping $X$ into itself.

(II) For $x$ fixed in $X$, $\phi(w, A)x$ is a continuous mapping of the closure of $S$ into $X$.

(III) For $x$ fixed in $X$, $\phi(w, A)x$ is an analytic mapping of $S$ into $X$.

(IV) For $x$ fixed in $X$, $\phi(w, A)x$ is a function in $H(X, D(A))$ satisfying

$$\|\phi(w, A)x\|_{H(X, D(A))} \leq \max\{K_0ML, K_1L(1 + M + TM)\}\|x\|.$$

Assuming for the time being I, II, III and IV above, it is clear that mapping $\Phi_s(A)$ of $X$ into $(X, D(A))_s$, defined by

$$\Phi_s(A)x = [\phi(w, A)x](s) = \phi(s, A)x$$
is the required bounded operator of our theorem since
(1) For $x_1$, $x_2$ in $X$ and $a_1$, $a_2$ scalars,
\[
\Phi_s(A)(a_1x_1 + a_2x_2) = \left[\phi(w, A)(a_1x_1 + a_2x_2)\right](s) = \phi(s, A)(a_1x_1 + a_2x_2) = a_1\phi(s, A)x_1 + a_2\phi(s, A)x_2 = a_1\Phi_s(A)x_1 + a_2\Phi_s(A)x_2.
\]
Thus, the operator $\Phi_s(A)$ is linear.

(2) For $x$ in $X$,
\[
\|\Phi_s(A)x\|_{H(D,A)} = \inf_{f(t) = \Phi_s(A)x} \{\|f\|_{H(D,A)}\}
\leq \|\phi(w, A)x\|_{H(D,A)}
\leq \max\{K_0ML, K_1L(1 + M + TM)\}\|x\|.
\]
Hence, $\Phi_s(A)$ is bounded with norm satisfying
\[
\|\Phi_s(A)\| \leq \max\{K_0ML, K_1L(1 + M + TM)\}.
\]

We proceed now to establish the validity of I, II, III and IV.
(1) Statement (I) is a direct consequence of assumptions (a), (b), part (2), and a classical result of complex functional analysis [3].

(2) To verify (II), we recall that $B(D)$, the boundary of $D$, is rectifiable with length $L$ and that $\|(z - A)^{-1}\|$ is continuous on $B(D)$ with maximum value $M$. Let $x$ in $X$ be fixed. Let $w$, $w_0$ be points of the closure of the open strip $S$. Then
\[
\|\phi(w, A)x - \phi(w_0, A)x\|
= \left\| \left( \int_{B(D)} \phi(w, z)(z - A)^{-1}dz \right)x - \left( \int_{B(D)} \phi(w_0, z)(z - A)^{-1}dz \right)x \right\|
= \left\| \left( \int_{B(D)} (\phi(w, z) - \phi(w_0, z))(z - A)^{-1}dz \right)x \right\|
\leq \left\{ \int_{B(D)} \|\phi(w, z) - \phi(w_0, z)\|\|(z - A)^{-1}\|\|dz\| \right\}\|x\|
\leq C\|w - w_0\|\left( \int_{B(D)} \|(z - A)^{-1}\|\|dz\| \right)\|x\|
\leq CML\|w - w_0\|\|x\|.
\]
(by the Lipschitz condition of assumption (b))
The continuity of $\phi(w, A)x$ on the closure of $S$ is now evident.
(3) We establish (III) as follows. In accordance with assumption (b), part (3), the partial derivative $\phi_w(w, z)$ exists for $w$ in $S$ and $z$ in $B(D)$, and is continuous in $z$ on $B(D)$ for each fixed $w$ in $S$. Therefore, $\int_{B(D)}\phi_w(w, z)(z-A)^{-1}dz$ exists.

Now let a positive number $\epsilon$ be given. Then for complex $h$ sufficiently small in absolute value one has

$$
\left\| \frac{(\phi(w+h, A)x-\phi(w, A)x)}{h} - \left( \int_{B(D)} \phi_w(w, z)(z-A)^{-1}dz \right)x \right\|
$$

$$
= \left\| \left\{ \int_{B(D)} \left\{ \frac{(\phi(w+h, z)-\phi(w, z)}{h}-\phi_w(w, z) \right\}(z-A)^{-1}dz \right\}x \right\|
$$

$$
\leq \left\{ \int_{B(D)} \frac{1}{h} \left\| (\phi(w+h, z)-\phi(w, z)) - \phi_w(w, z) \right\| (z-A)^{-1}dz \right\} \|x\|
$$

$$
\leq \left\{ \int_{B(D)} \epsilon \| (z-A)^{-1} \| dz \right\} \|x\| \quad \text{(by assumption (b), part (3))}
$$

$$
\leq \epsilon M L \|x\|.
$$

The analyticity of $\phi(w, A)x$ in $S$ is now clear.

(4) Given the previous results concerning the continuity and analyticity of $\phi(w, A)x$, one is able to show that $\phi(w, A)x$ is a member of $H(X, D(A))$ by ascertaining the following:

(i) For $t$ real, $\phi(1+it, A)x$ is a member of $D(A)$.

(ii) There exist constants $M_0, M_1$ such that

$$
\|\phi(1+it, A)x\|_{D(A)} \leq M_1 \quad \text{for all real } t, \quad \text{and}
$$

$$
\|\phi(it, A)x\| \leq M_0 \quad \text{for all real } t.
$$

Let $t$ be a fixed real number and let $x$ be a fixed member of $X$. Let $y=\phi(1+it, A)x$. Then $y=(\int_{B(D)}\phi(1+it, z)(z-A)^{-1}dz)x$. Thus there exists a sequence of partitions $P_n$ of $B(D)$ with “norms” converging to 0, and a corresponding sequence of “Riemann sums”,

$$
S_n = \sum_{k=1}^{n} \phi(1+it, z'_k)(z'_k - A)^{-1}(z_k - z_{k-1})
$$

satisfying $S_n x$ converges to $y$ as $n$ approaches infinity.

One observes that for each $x$ in $X$, $S_n x$ is an element of $D(A)$. The operator $A$ by assumption is closed in $X$. Thus, to establish that $y$ is a member of $D(A)$, it is sufficient to show that the sequence $A(S_n x)$ has a limit as $n$ approaches infinity. We presently obtain this
result. Recall that \( B(D) \), the boundary of \( D \) is compact. Thus the set,

\[
\{ |z| \mid z \text{ is a member of } B(D) \}
\]

has a maximum value which we denote by "\( T \)". Now let \( z \in B(D) \). Then

\[
\| A(z - A)^{-1} \| = \| ((z - A) + z)(z - A)^{-1} \|
\leq \| (z - A)(z - A)^{-1} \| + |z| \| (z - A)^{-1} \|
= \| I \| + |z| \| (z - A)^{-1} \| \quad (I \text{ is the identity operator})
\leq 1 + TM.
\]

Thus, for each \( z \) in \( B(D) \), \( A(z - A)^{-1} \) is a bounded linear operator on \( X \). Furthermore, \( A(z - A)^{-1} \) is continuous on \( B(D) \), for let \( z, z_0 \) be members of \( B(D) \). Then

\[
\| A(z - A)^{-1} - A(z_0 - A)^{-1} \| = \| A((z - A)^{-1} - (z_0 - A)^{-1}) \|
\leq \| (z_0 - z)(z - A)^{-1} \| \| (z_0 - A)^{-1} \|
\leq |z_0 - z| \| (1 + TM)M \|.
\]

By assumption (b), part (2) \( \phi(1 + it, z) \) is an analytic function of \( z \) in a neighborhood of the closure of \( D \). Hence, it is certainly continuous on \( B(D) \). Thus \( \int_{B(D)} \phi(1 + it, z) A(z - A)^{-1} dz \) exists. Consequently,

\[
A(S_n x) = \left\{ \sum_{k=1}^{n} \phi(1 + it, z_k') A(z_k' - A)^{-1}(z_k - z_{k-1}) \right\} x
\]

converges to \( \left\{ \int_{B(D)} \phi(1 + it, z) A(z - A)^{-1} dz \right\} x \) as \( n \) approaches infinity. Thus, \( y \in D(A) \).

One now observes that

\[
\| \phi(1 + it, A)x \| = \| \left\{ \int_{B(D)} \phi(1 + it, z)(z - A)^{-1} dz \right\} x \|
\leq \left\{ \int_{B(D)} | \phi(1 + it, z) | \| (z - A)^{-1} \| |dz| \right\} \| x \|
\leq \left\{ \int_{B(D)} K_1 \| (z - A)^{-1} \| |dz| \right\} \| x \|
\leq K_1 ML \| x \|.
\]

(by assumption (b), part (4))
Thus,

\[ \| \phi(1 + it, A)x \|_{D(A)} = \| \phi(1 + it, A)x \| + \| A(\phi(1 + it, A)x) \| \]

\[ = \left\| \left\{ \int_{B(D)} \phi(1 + it, z)(z - A)^{-1}dz \right\} x \right\| \]

\[ + \left\| \left\{ \int_{B(D)} \phi(1 + it, z)A(z - A)^{-1}dz \right\} x \right\| \]

\[ \leq K_1ML\|x\| + K_1(1 + TM)L\|x\| \]

\[ = \{K_1ML + K_1(1 + TM)L\}\|x\| \]

\[ = \{K_1L(1 + M + TM)\}\|x\|. \]

Similarly,

\[ \| \phi(it, A)x \| = \left\| \left\{ \int_{B(D)} \phi(it, z)(z - A)^{-1}dz \right\} x \right\| \]

\[ \leq \left\{ \int_{B(D)} |\phi(it, z)| \| (z - A)^{-1} \| dz \right\} \| x \| \]

\[ \leq \left\{ \int_{B(D)} K_0\| (z - A)^{-1} \| dz \right\} \| x \| \]

(by assumption (b), part (4))

\[ \leq K_0ML\|x\|. \]

Thus, \( \phi(w, A)x \) is a member of \( H(X, D(A)) \), and

\[ \| \phi(w, A)x \|_{H(X, D(A))} = \max \left\{ \sup_{t} \| \phi(it, A)x \|, \sup_{t} \| \phi(1 + it, A)x \|_{D(A)} \right\} \]

\[ \leq \max \{ (K_0ML)\|x\|, K_1L(1 + M + TM)\|x\| \} \]

\[ = \max \{ K_0ML, K_1L(1 + M + TM)\}\|x\|. \]

**Bibliography**


Rutgers University, New Brunswick, New Jersey 08903