THE HYPERQUASICENTER OF A FINITE GROUP. I

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ABSTRACT. The quasicenter is a generalisation of the idea of the center of a group and this has been used to define the hyperquasicenter. The concept that is essentially involved is that of quasinormality. The quasicenter has been shown to be nilpotent and the hyperquasicenter has been identified as the largest supersolvably immersed subgroup.

Introduction. The concept of a quasicentral element of a group G is generalisation of the idea of an element in the center. An element $g\in G$ is called a quasicentral element (QC-element) of G if $\langle g\rangle\langle x\rangle = \langle x\rangle\langle g\rangle \forall x\in G$. This leads to the definition of the quasicenter $Q(G)$ which is the subgroup generated by all quasicentral elements and it can be easily shown to be a characteristic subgroup. The principal results related to quasicentral elements and the quasicenter are: (1) If $x\in G$ is a QC-element then x^r is a QC-element for all integers r. (2) $Q(G)$ is nilpotent.

The quasicenter leads to the definition of the hyperquasicenter in the following manner. Let $Q_0=1$ and Q_{i+1}/Q_i be the quasicenter of G/Q_i. This gives an ascending chain of characteristic subgroups as follows: $1=Q_0\subset Q_1\subset Q_2\subset \cdots \subset Q_n=Q^*$. The terminal member Q^* of this series (termed upper quasicentral series) is called the hyperquasicenter of G. The principal result described in the paper is that the hyperquasicenter has been identified as the largest supersolvably immersed subgroup of G.

DEFINITION 1.1. An element $x\in G$ is a quasicentral element (QC-element) iff $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all y in G.

THEOREM 1.2. *If x is a QC-element of a group G then x^r is a QC-element for all integers r.*

PROOF. Let y be an arbitrary but fixed element of G. Then we must prove $\langle x^r\rangle\langle y\rangle = \langle y\rangle\langle x^r\rangle$ for all integers r. We argue by induction on $|G|$. Thus we can assume that $G=\langle x, y\rangle$. Set $X=\langle x\rangle$ and $Y=\langle y\rangle$.

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Then $G=XY$. There are two cases. First consider that either X or Y contains a nontrivial normal subgroup Z of G. Set $\overline{G}=G/Z$, $\overline{X}=XZ/Z$ and $\overline{Y}=YZ/Z$. By induction $\overline{X}^r\overline{Y}=\overline{Y}\overline{X}^r$. So $X^rYZ=YX^rZ$. By the assumption on Z, $X^rY=YX^r$ and x^r is a QC-element in this case. In the remaining case neither X nor Y contains a normal subgroup of G. Hence $X\cap Y=1$. Then $|G|=|X|\cdot|Y|$.

Suppose that $|Y|\geqq|X|$. Then as Y is not normal in G, G has a double coset of the form $YxY\neq Yx$. Then

$$YxY = |Y|^2/|Y\cap Y^{x^{-1}}|.$$

Set $Y_0=Y\cap Y^{x^{-1}}$. Then $Y_0^x\subseteq Y$. As Y is cyclic, $Y_0^x=Y_0$. This implies that $Y_0\subset G=XY$. Hence $Y_0=1$. Now we have $|G|>|YxY|=|Y|^2/|Y_0|=|Y|^2\geqq|Y|\cdot|X|=G$, a contradiction. The same argument applies if $|X|\geqq|Y|$. Hence the theorem follows.

Definition 1.3. The quasicenter $Q(G)$ of a group G is the subgroup generated by all QC-elements of G.

The following example shows that every element of $Q(G)$ need not be a QC-element and also that the product of any two QC-elements is not necessarily a QC-element.

Example. Let $G=\langle a, b\rangle\langle x\rangle=\langle x\rangle\langle a, b\rangle$ where $a^3=1$, $b^3=1$, $ab=ba$, $a^x=a^2$, $b^x=b$, $x^2=1$. $Q(G)=\langle a, b\rangle$ and a, b are QC-elements of G. But $ab\in Q(G)$ is not a QC-element of G.

The structure of $Q(G)$ is investigated next and the following lemma is necessary for that.

Lemma 1.4. *Every Sylow subgroup of the quasicenter $Q(G)$ of a group G is generated by QC-elements of G.*

Proof. Let $\{x_1, x_2, \cdots, x_t\}=X$ be the set of all QC-elements of G and note that $\langle X\rangle=Q(G)$. Each x_i can be written as $x_i=y_iz_i=z_iy_i$, where y_i is a p-element, z_i is a p'-element, $i=1, 2, \cdots, t$, and y_i, z_i are QC-elements of G being powers of x_i. Define $X_p=\{y_1, y_2, \cdots, y_t\}$ and let θ be any automorphism of G. The image of the QC-element x_i under θ is $y_i^\theta z_i^\theta$ and it is a QC-element of G. Therefore $y_i^\theta z_i^\theta$ is in X and y_i^θ is in X_p. Consequently X_p is a normal set. Now $y_i\forall i$ being a QC-element of G it follows that $\langle X_p\rangle$ is a p-group and therefore if P is a Sylow p-subgroup of $Q(G)$ then $\langle X_p\rangle\subseteq P$. But $Q(G)=\langle x_1\rangle\cdot\langle x_2\rangle\cdot\cdots\cdot\langle x_t\rangle=\langle y_1\rangle\cdot\langle y_2\rangle\cdot\cdots\cdot\langle y_t\rangle\cdot\langle z_1\rangle\cdot\langle z_2\rangle\cdot\cdots\cdot\langle z_t\rangle$. Hence $|P|=|\langle y_1\rangle\cdot\langle y_2\rangle\cdot\cdots\cdot\langle y_t\rangle|$ and so $P=\langle X_p\rangle$. The assertion in the lemma therefore follows.

Theorem 1.5. *The quasicenter of a group G is nilpotent.*

PROOF. Let P_1 and P_2 be two Sylow subgroups of $Q(G)$, corresponding to the prime divisor p of $Q(G)$. Let $P_1 = \langle g_1, g_2, \cdots, g_n\rangle$, where g_i is a QC-element of G for $i = 1, 2, \cdots, n$. If $g_i \notin P_2$ for some i, then consider $P_2\langle g_i\rangle = \langle g_i\rangle P_2$. It is a p-group whose order is larger than $|P_2|$ a contradiction. Hence $P_1 \subseteq P_2$ and so $P_1 = P_2$, since $|P_1| = |P_2|$. Thus there is a unique Sylow subgroup of $Q(G)$ corresponding to each prime divisor of $Q(G)$.

Hence $Q(G)$ is nilpotent. If $H \lhd G$ then $Q(H)$ in general neither contains nor is included in $Q(G)$ always. For $G = A_4$ and $H \lhd G$, $Q(H) \not\subseteq Q(G)$ and if $G =$ the quaternion group, $H \lhd G$ then $Q(G) \not\subseteq Q(H)$. This leads to the investigation of the relationship of the quasicenter of a group G and quasicenters of subgroups with additional restrictions besides normality imposed on the subgroups.

THEOREM 1.6. *If a group G is the direct product of subgroups H and K then $Q(G) \subseteq Q(H) \times Q(K)$. Further, if $(|H| \cdot |K|) = 1$ then $Q(G) = Q(H) \times Q(K)$.*

REMARK. If $G = H \times K$ it is not in general true that a QC-element of H or K will be a QC-element of G. That is why the equality of $Q(G)$ and $Q(H) \times Q(K)$ in general seems improbable. The following example confirms this assertion.

EXAMPLE. Let $G = \langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$, where $|a| = 3$, $|b| = 9$, $a^{-1}ba = b^4$. Note that 'a' is a QC-element. Now if $S = \langle a_1\rangle \cdot \langle b_1\rangle \times \langle a_2\rangle \cdot \langle b_2\rangle = G_1 \times G_2$ where G_1 and G_2 are copies of G then a_1 is not a QC-element of S since $\langle a_1\rangle$ does not permute with $\langle b_1 b_2\rangle$.

The definition of the hyperquasicenter follows from the definition of the quasicenter in an exactly identical manner as that of the hypercenter follows from the definition of the center. Let G be a group and suppose $Q_0 = 1$ and Q_{i+1}/Q_i is the quasicenter of G/Q_i. Then the terminal member Q^* of the chain $1 = Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = Q^*$ of characteristic subgroups is called the hyperquasicenter of G. We shall now prove the main result of the paper. For the sake of completeness we include the following definition.

DEFINITION 1.7. A subgroup M of a group G is called supersolvably immersed in G if for each homomorphism of G the image of M contains a cyclic subgroup which is normal in the homomorphic image of G.

THEOREM 1.8. *The hyperquasicenter of a group G is the largest supersolvably immersed subgroup of G.*

PROOF. Let Q^* be the hyperquasicenter of G. We prove first that every supersolvably immersed subgroup of G is contained in Q^* and this we do by induction on the order of G. For $|G| = 1$ this is true and assume it to be true for all groups of order less than $|G|$. If M is a supersolvably immersed subgroup of G then it shall contain a cyclic subgroup $\langle a \rangle$ which is normal in G and note that $M/\langle a \rangle$ is a supersolvably immersed subgroup of $G/\langle a \rangle$. Since $\langle a \rangle$ is a normal subgroup of G it follows that $\langle a \rangle\langle x \rangle = \langle x \rangle\langle a \rangle \forall x \in G$. Therefore '$a$' is a QC-element of G and $\langle a \rangle \subseteq Q^*$. Hence by induction $M/\langle a \rangle \subseteq Q^*/\langle a \rangle$. (It is not difficult to show that if T is a normal subgroup contained in the hyperquasicenter Q^* of a group G then the hyperquasicenter of G/T equals Q^*/T.) Thus $M \subseteq Q^*$.

We now prove that the hyperquasicenter is a supersolvably immersed subgroup of G and the main step in proving this is to show that the quasicenter is supersolvably immersed as in that case supersolvable immersion of the hyperquasicenter will immediately follow.

Let P be a Sylow p-subgroup of the quasicenter $Q(G)$ of G. P is generated by QC-elements and let x be a QC-generator of P. Suppose y is a p-element of G. Set $X = \langle x \rangle$ and $Y = \langle y \rangle$. Then $XY = YX$ by assumption and so XY is a group. Therefore $x^y = x^a y^b$ for some integers a and b. But $y^b = x^{-a}x^y$ is a p-element belonging to P. Thus $y^b = 1$ and so $x^y = x^a$. X is therefore normalised by all p-elements of G. But from the structure of the automorphism group of a cyclic p-group it now follows that X is centralised by y^{p-1} where y is any p-element of G. Since P is generated by QC-elements the above statement implies that $[P/D(P), y^{p-1}] = 1$ for all p-elements of G where $D(P)$ is the Frattini subgroup of P. From this it follows that $[P, y^{p-1}] = 1$. By Theorem 6.1 of Baer in [6] it follows therefore that P is supersolvably immersed and so the quasicenter $Q(G)$ is also supersolvably immersed. The assertion in the theorem therefore follows.

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