A REGULAR LINDELÖF SEMIMETRIC SPACE WHICH HAS NO COUNTABLE NETWORK

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ABSTRACT. A completely regular semimetric space M is constructed which has no σ-discrete network. The space M constructed has the property that every subset of M of cardinality $2^{\aleph_0}$ contains a limit point of itself; thus, assuming $2^{\aleph_0}=\aleph_1$, M is Lindelöf. It is also shown from the same space M that, assuming $2^{\aleph_0}=\aleph_1$, there exists a regular Lindelöf semimetric space X such that $X\times X$ is not normal (hence not Lindelöf).

A. V. Arhangel'skiĭ asks the following question [1, p. 131]: Does every symmetric space have a σ-discrete network? R. W. Heath asks in [3]: Does every regular Lindelöf semimetric space have a countable network? In this paper Arhangel'skiĭ's question is answered negatively by the construction of a completely regular semimetric (hence symmetric) space M which has no σ-discrete network. In that same space M, every subset of cardinality $2^{\aleph_0}$ contains a limit point of itself, so that, assuming the continuum hypothesis, M would be Lindelöf. Thus, subject to the continuum hypothesis, Heath's question is also answered negatively. Finally it is shown from the same space M that, assuming the continuum hypothesis, there exists a regular Lindelöf semimetric space X such that $X\times X$ is not Lindelöf—in fact not even normal.

A collection of subsets $\mathcal{A}$ of a topological space X is said to be a *network* for X if, given an open set U containing a point x, there exists $A\in\mathcal{A}$ such that $x\in A\subseteq U$. A network $\mathcal{A}$ is a *σ-discrete network* if $\mathcal{A}$ is the countable union of discrete collections. A topological space X is said to be a *semimetric space* if there exists a real valued, nonnegative, symmetric function d on $X\times X$ such that:

a. $d(x, y)=0$ iff $x=y$,

b. $x\in\overline{A}$ iff $\mathrm{glb}\{d(x, y)\mid y\in A\}=0$.

Let I denote the unit interval. Let $\Delta=\{(x, x)\mid x\in I\}$. If A is a set, let card A denote the cardinality of A. Let Γ denote the first ordinal such that card $\{\alpha\mid\alpha \text{ an ordinal}, \alpha<\Gamma\}=2^{\aleph_0}$.

THEOREM 1. *There is a completely regular semimetric* (*hence symmetric*) *space M such that* (i) *M has no σ-discrete network*, (ii) *every*

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subset of M of cardinality $2^{\aleph_0}$ contains a limit point of itself so that, if $\aleph_1 = 2^{\aleph_0}$, M is Lindelöf.

THEOREM 2. *If $\aleph_1 = 2^{\aleph_0}$, then there is a regular Lindelöf semimetric space X such that $X \times X$ is not normal (hence not Lindelöf).*

LEMMA. *There exists a subset V of I such that:*

1. *If U is an open subset of I, then* $\text{card}(U \cap V) = 2^{\aleph_0}$.
2. *There exists no pair f and D such that:*

(i) $D \subseteq V$ *and* card $D = 2^{\aleph_0}$,

(ii) $f: D \to V$,

(iii) *either f is strictly increasing and $\{(x, f(x)) \mid x \in D\}$ is bounded away from Δ, or f is strictly decreasing.*

PROOF OF LEMMA. Let $\mathfrak{C}$ denote the set of all closed subsets of I. If $A \in \mathfrak{C}$, let $\mathfrak{F}(A)$ denote the set of all monotone functions from A into I such that point-inverses are at most countable. Let

$$\mathfrak{F} = \bigcup \{\mathfrak{F}(A) \mid A \in \mathfrak{C}\}.$$

It is easily shown that $\text{card}(\mathfrak{F}) = 2^{\aleph_0}$. Let $\mathfrak{U}$ denote the set of all open subsets of I. Let $\mathfrak{U} = \{\mathfrak{U}_\alpha : \alpha < \Gamma\}$ and $\mathfrak{F} = \{f_\alpha : \alpha < \Gamma\}$ be well-orderings of $\mathfrak{U}$ and $\mathfrak{F}$, recalling that Γ is the first ordinal of cardinal $2^{\aleph_0}$. Define a set V as follows. Select a point x_1 from U_1. For each β, $1 < \beta < \Gamma$, such that x_α is defined for all $\alpha < \beta$, pick x_β to be some point of U_β such that

$$x_\beta \notin [\bigcup \{\overset{-1}{f_\alpha}(x_\tau) \mid \alpha \leqq \beta, \tau < \beta\} \cup \{x_\tau \mid \tau < \beta\} \cup \{f_\alpha(x_\tau) \mid \alpha \leqq \beta, \tau < \beta\}].$$

Let $V = \{x_\alpha \mid \alpha < \Gamma\}$. The proof that V satisfies condition 1 is obvious. To see that condition 2 is satisfied, suppose not; then there exists a pair f and D satisfying 2(i), 2(ii), and 2(iii). Assume f is strictly decreasing (the argument is similar if f is strictly increasing and $\{(x, f(x)) \mid x \in D\}$ is bounded away from Δ). Extend f to $\overline{D}$ (call the extension $\hat{f}$) as follows: if $t \in \overline{D}$, then

$$\begin{aligned} \hat{f}(t) &= f(t) && \text{if } t \in D, \\ &= \operatorname*{lub}_{x > t; x \in D} f(x) && \text{if } t \text{ is a limit point of } D \text{ from above and } t \notin D, \\ &= \operatorname*{glb}_{x < t; x \in D} f(x) && \text{if } t \text{ is not a limit point of } D \text{ from above and } t \notin D. \end{aligned}$$

It is easily checked that $\hat{f} \in \mathfrak{F}$. Therefore, for some ordinal α, $\hat{f} = f_\alpha$. Now f_α has the property that there exists at most one x_θ in D such that $f_\alpha(x_\theta) = x_\theta$ (in the case of f_α strictly increasing etc., $\{(x, f(x)) \mid x \in D\}$ being bounded away from Δ guarantees for every θ, $\theta < \Gamma$, $f_\alpha(x_\theta) \neq x_\theta$).

Now there exist x_γ, $x_\beta \in D$ such that γ, $\beta > \max\{\theta, \alpha\}$, $\gamma \neq \beta$, and $f_\alpha(x_\gamma) = x_\beta$. For if not, then f_α is a one-to-one correspondence between $\{x_\tau | \tau > \max(\theta, \alpha)\}$ and $\{x_\tau | \tau \leqq \max(\theta, \alpha)\}$, which is a contradiction since $\mathrm{card}[\{x_\tau | \tau > \max(\theta, \alpha)\}] > \mathrm{card}[\{x_\tau | \tau \leqq \max(\theta, \alpha)\}]$. Thus, let x_γ, x_β be two such points. If $\gamma < \beta$, then we have a contradiction to the definition of V. If $\gamma > \beta$, as $x_\gamma \in f_\alpha^{-1}(x_\beta)$, we also have a contradiction to the definition of V. Thus, the proof is complete.

PROOF OF THEOREM 1. Let V be as in the Lemma. Choose M to be a subset of $V \times V - \Delta$ such that: (1) if U is open in $I \times I$, then $\mathrm{card}(U \cap M)$ is $2^{\aleph_0}$, and (2) if Π_1 and Π_2 denote the two projection functions on $V \times V$ into V, then for each $m \in M$, $\Pi_i(m) \notin \Pi_i(M - \{m\})$, $i = 1, 2$. Let d denote the usual Euclidean metric of $I \times I$. Define d^* on $M \times M$ as follows: if (a, b), $(c, d) \in M$, then

$$\begin{aligned} d^*[(a, b), (c, d)] &= d[(a, b), (c, d)] \quad \text{if } a \leqq c \text{ and } b \leqq d \text{ or } a \geqq c \text{ and } b \geqq d, \\ &= 1 \quad \text{if } a < c \text{ and } b > d \text{ or } a > c \text{ and } b < d. \end{aligned}$$

It is easily shown that d^* generates a completely regular semimetric topology on M (denoted by $M(d^*)$) with semimetric d^* by defining: A is closed in M iff $A \supseteq \{x \in M | d^*(x, A) = 0\}$. We now show that M (with the $M(d^*)$ topology) has the property that every subset A of cardinality $2^{\aleph_0}$ has a limit point in A. Let A be a subset of M such that card A is $2^{\aleph_0}$. Suppose A is discrete in itself. Then there exists a subset A' of A such that card $A' = 2^{\aleph_0}$ and a positive number δ such that (c, d), $(a, b) \in A'$ implies $d^*[(a, b), (c, d)] > \delta$. Let $(p, q) \in A'$ such that for each Euclidean disk S about (p, q), $\mathrm{card}(S \cap A') = 2^{\aleph_0}$. Let S be a Euclidean disk of radius less than $\delta/2$ about (p, q). If (a, b), $(c, d) \in S \cap A'$ and $a < c$, then $b > d$. For if not, then $a < c$ and $b < d$ implies $d^*[(a, b), (c, d)] = d[(a, b), (c, d)] < \delta$, a contradiction. Note $S \cap A'$ is a strictly decreasing function on $\Pi_1(S \cap A')$ into V and $\mathrm{card}[\Pi_1(S \cap A')] = 2^{\aleph_0}$, a contradiction to the definition of V. Hence part (ii) of Theorem 1 is true. We now show that M has no σ-discrete network. Suppose it does, say $\mathfrak{A}$ where $\mathfrak{A} = \bigcup\{B_i | i = 1, 2, \cdots\}$. By part (ii) of Theorem 1, card $B_i < 2^{\aleph_0}$, hence card $\mathfrak{A} < 2^{\aleph_0}$. Let $N(x, y)$ denote the d^* sphere of radius $\frac{1}{2}$ about (x, y). There exist a subset M^1 of M and an A in $\mathfrak{A}$ such that $\mathrm{card}(M^1) = 2^{\aleph_0}$ and $(a, b) \in M^1$ implies $(a, b) \in A \subseteq N(a, b)$. Let $(p, q) \in M^1$ such that, if S is a Euclidean disk about (p, q), then $\mathrm{card}(S \cap M^1) = 2^{\aleph_0}$. Let S be a Euclidean disk of radius r about (p, q) such that $0 < r < \min\{\frac{1}{4}, \frac{1}{2}d[(p, q), \Delta]\}$. Let (a, b), $(c, d) \in S \cap M^1$. If $a < c$, then $b < d$. For if not, there exists points (a, b) and (c, d) in $S \cap M^1$ such that $a < c$ and $b > d$. Then $(c, d) \notin N(a, b)$, but $(c, d) \in A \subseteq N(a, b)$ which is a contradiction. Thus

$S \cap M^1$ is a strictly increasing function on $\Pi_1(S \cap M^1)$ into V such that $d[S \cap M^1, \Delta] > 0$ and card $\Pi_1(S \cap M^1) = 2^{\aleph_0}$, which is a contradiction to the definition of V. Hence M (with the $M(d^*)$ topology) has no σ-discrete network.

PROOF OF THEOREM 2. Define d' on $M \times M$ by

$$\begin{aligned} d'[(a, b), (c, d)] &= d[(a, b), (c, d)] \quad \text{if } a \leqq c \text{ and } b \geqq d \text{ or } a \geqq c \text{ and } b \leqq d, \\ &= 1 \quad \text{if } a < c \text{ and } b > d \text{ or } a > c \text{ and } b < d. \end{aligned}$$

Then, again, we have a regular Lindelöf semimetric space (with semimetric d') which has no countable network (similar arguments as in the proof of Theorem 1). Denote the d' semimetric topology by $M(d')$. Let X_1 (respectively X_2) denote M with the $M(d^*)$ (respectively $M(d')$) topology. Let X be the free union [2] of X_1 and X_2 (i.e., $X = \{(i, z) : z \in M, i = 1, 2\}$ with the topology generated by $\{\{i\} \times U : U$ is open in X_i, $i = 1, 2\}$). Then X is a regular Lindelöf semimetric space which has no countable network and $X \times X$ is not normal. To see $X \times X$ is not normal (and hence not Lindelöf), consider the set $R = \{((1, z), (2, z)) : z \in M\}$ which is a subset of $X \times X$. It is easily shown that R is closed in $X \times X$. We now show that R is discrete in itself. Let $z \in M$. Let $((1, z), (2, z)) \in R$. Let S_1 denote the d^* semimetric sphere of radius $\frac{1}{2}$ about z, S_2 denote the d' semimetric sphere of radius $\frac{1}{2}$ about z, $S_1^1 = \{1\} \times S_1$, and $S_2^1 = \{2\} \times S_2$. Let $z = (x, y)$. Suppose $((1, c), (2, c)) \in R \cap (S_1' \times S_2')$. If $c = (a, b)$, then $a \leqq x$ and $b \leqq y$ or $a \geqq x$ and $b \geqq y$ as $(1, c) \in S_1'$. Similarly $a \leqq x$ and $y \leqq b$ or $a \geqq x$ and $y \geqq b$ as $(1, c) \in S_2'$. Hence $a = x$ and $b = y$. Therefore $R \cap (S_1' \times S_2') = \{((1, z), (2, z))\}$ and R is discrete in itself. Note $X \times X$ is separable. Thus, by Theorem 1 of [4], $X \times X$ is not normal.

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