BOUNDED IN THE MEAN SOLUTIONS OF $\Delta u = Pu$ ON RIEMANNIAN MANIFOLDS

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ABSTRACT. Let Φ be a convex positive increasing function and $d = \lim_{t\to\infty} \Phi(t)/t$. A harmonic function u on a Riemann surface R is called Φ-bounded if $\Phi(|u|)$ is majorized by a harmonic function on R. M. Parreau has shown that if $d < \infty$ ($d = \infty$, resp.), then every positive (bounded, resp.) harmonic function on R reduces to a constant if and only if every Φ-bounded harmonic function does. In this paper analogues of these results are given for the equation $\Delta u = Pu$ ($P \geqq 0$) on a Riemannian manifold.

Let R be a noncompact orientable Riemannian n-manifold, P a nonnegative smooth n-form on R and Φ a convex positive increasing function on $[0, \infty)$ with $\Phi(0) = 0$. A solution u of $\Delta u = Pu$ is said to be Φ-bounded or bounded in the mean on R if $\Phi(|u|)$ has a solution majorant on R. Set $d = \lim_{t\to\infty} \Phi(t)/t$. Denote by $PX(R)$ the subset of solutions of $\Delta u = Pu$ on R satisfying a boundedness property X; examples of X are N (nonnegative), B (bounded), Φ (Φ-bounded). The maximum number of linearly independent functions in $PX(R)$ will be denoted by $\dim PX(R)$ even though $PX(R)$ in general is not a vector space. Consider the following statements:

(i) *If* $d < \infty$, *then* $\dim PN(R) = i$ *if and only if* $\dim P\Phi(R) = i$.
(ii) *If* $d = \infty$, *then* $\dim PB(R) = i$ *if and only if* $\dim P\Phi(R) = i$.

Parreau [6] established the validity of (i) and (ii) in the case $P \equiv 0$ and $i = 1$, that is for harmonic functions. In this paper we consider the case $P \not\equiv 0$. If $P \not\equiv 0$, the constants are not solutions of $\Delta u = Pu$ and therefore degeneracy of a Riemannian manifold R with respect to a property X has been taken to be $\dim PX(R) = 0$. In view of Myrberg's [3] result that $\dim PN(R)$ is always greater than zero and the fact that when $\dim PB(R) \geqq 1$ a one dimensional subspace plays virtually the same role as the constants do in the space of bounded harmonic functions, we shall consider two degrees of degeneracy $\dim PX(R) = i$, $i = 0$ and 1.

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Our approach invokes some recent results in the axiomatic theory of harmonic functions. In particular we begin by showing that an easy consequence of Loeb-Walsh [3] is the two domain criterion [1, p. 213], well known for harmonic functions on Riemann surfaces. Nakai [5] has shown that Parreau's results are valid for almost arbitrary nonnegative real-valued functions Φ but in our extension to $P\neq 0$ we have been unable to relax the original requirements on Φ.

1. We first introduce some terminology and results from Loeb-Walsh [2]. Let $\mathcal{H}$ be a harmonic class on W satisfying Loeb's Axiom IV, i.e. $1\in\overline{\mathcal{H}}_W$. Let e be the greatest $\mathcal{H}$-harmonic minorant of 1 and assume $e\neq 0$. It is also assumed that the Banach lattice $\mathcal{BH}_W$ of bounded functions in $\mathcal{H}_W$ does not reduce to the constants. (Note that in the case $\Delta u=Pu$, $P\neq 0$, this already follows from the fact that $e\neq 1$.) Let W^* be the $\mathcal{BH}_W$-compactification of W, $\Gamma=W^*-W$ and

$$\Delta = \{t\in\Gamma \mid e(t) = 1 \text{ and } f\wedge_{\mathcal{H}} g(t) = f\wedge g(t) \text{ for every } f,\, g\in\mathcal{BH}_W\}.$$

The symbols $\overline{A}$, bA will denote the closure and boundary of A with respect to $\overline{W}=W\cup\Delta$ and by ∂A we mean $bA\cap W$. By 1.1 and 2.1 of [2] we see that for any region $\Omega\subset W$, $b\Omega$ is associated to $\overline{\mathcal{H}}^b_\Omega$ (i.e. if $s\in\overline{\mathcal{H}}_\Omega$, is bounded from below and $\lim s\geqq 0$ at $b\Omega$, then $s\geqq 0$). Theorem 2.3 of [2] states that the restriction mapping gives an isometric isomorphism of $\mathcal{BH}_W$ onto $C(\Delta)$ which preserves positivity and the lattice operations. From 1.2 and 2.5 of [2] we see that every point of Δ is regular for the Dirichlet problem.

We now introduce the

DEFINITION. *A region $\Omega\subset W$ is in the class $SO_{\mathcal{BH}}$ if $\partial\Omega$ is regular for the Dirichlet problem and there exists no nonzero element of $\mathcal{BH}_\Omega$ which vanishes continuously on $\partial\Omega$.*

As a consequence we have the

THEOREM. *There exist at least k $(\geqq 1)$ pairwise disjoint regions $\Omega_i\subset W$, $\partial\Omega_i$ regular and $\Omega_i\notin SO_{\mathcal{BH}}$ if and only if* $\dim\mathcal{BH}_W\geqq k$.

If $\Omega_i\notin SO_{\mathcal{BH}}$, then $b\Omega_i-\partial\Omega_i\neq\emptyset$ because we know $b\Omega_i$ is associated to $\overline{\mathcal{H}}^b_\Omega$. Thus the existence of k such Ω_i's means that Δ contains at least k points and hence $\dim\mathcal{BH}_W=\dim C(\Delta)\geqq k$. Conversely suppose $\dim\mathcal{BH}_W\geqq k$. Then there exist k points $p_j\in\Delta$ and $u_i\in\mathcal{BH}_W$, $0\leqq u_i\leqq 1$ with the property $u_i(p_j)=\delta_{ij}$, $1\leqq i, j\leqq k$. Set $\Omega_i=\{x\in W \mid u_i > \max(u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_k)\}$. If $x_0\in\partial\Omega_i$, then there is a u_l such that $u_i(x_0)=u_l(x_0)$ and consequently u_i-u_l is a barrier for Ω_i at x_0. Thus $\partial\Omega_i$ is regular and, in view of the regularity of Δ, $b\Omega_i$ is regular. Since $b\Omega_i-\partial\Omega_i\neq\emptyset$ we can find $h\in\mathcal{BH}_{\Omega_i}$ such that $h\neq 0$ and $h|\partial\Omega_i=0$, i.e. $\Omega_i\notin SO_{\mathcal{BH}}$.

2. We now turn to the harmonic class determined by the solutions of $\Delta u = Pu$ on a Riemannian manifold R.

THEOREM. *If* $P \not\equiv 0$, *then* (i) *is valid for* $i = 0$ *and* 1.

If $d < \infty$, then there is a $c > 0$ such that $\Phi(t) \leqq ct$, $t \geqq 0$. By Myrberg's result we always have a nonzero solution $u \in PN(R)$ and since $\Phi(u) \leqq cu$, $u \in P\Phi(R)$. Thus (i) holds for $i = 0$.

Suppose dim $PN(R) > 1$ then by the above observation we also have dim $P\Phi(R) > 1$. That dim $P\Phi(R) > 1$ implies dim $PN(R) > 1$ is a direct consequence of a result of Naim [**4**, Corollary 10.2] to the effect that every Φ-bounded solution is the difference of two nonnegative ones.

3. Another result of Naim [**4**, Lemma 9] is that if u is a solution, then $\Phi(|u|)$ is a subsolution. From this we obtain the

LEMMA. $PB(\Omega) \subset P\Phi(\Omega)$, *for any* Φ.

In fact $\Phi(|u|)$ is a bounded subsolution for a function $u \in PB(\Omega)$. The family F of all supersolutions greater than $\Phi(|u|)$ is a nonempty Perron family. Thus the pointwise inf v of F is a solution with $\Phi(|u|) \leqq v$.

4. As in §1 we denote by SO_{PX} the subregions Ω of R such that $\partial\Omega$ is regular for the Dirichlet problem and $PX(\Omega)$ contains no nonzero element which vanishes continuously on $\partial\Omega$. The remainder of our result is based on the following (cf. [**1**, p. 218]).

THEOREM. *If* $d = \infty$, *then* $SO_{P\Phi} = SO_{PB}$, *otherwise* $SO_{P\Phi} = SO_{PN}$.

Myrberg's [**3**] argument can be used to show that $SO_{PN} = \emptyset$. Take $\Omega \subset R$ such that $\partial\Omega$ is regular and $\{R_n\}$ is an exhaustion of R, with ∂R_n regular and $\bar{R}_n$ compact in R. Fix $x \in \Omega \cap R_1$ and let ω_n be a solution on $\Omega \cap R_n$ such that $\omega_n | \partial\Omega \cap R_n = 0$, $\omega_n | \partial R_n \cap \Omega = c_n$ (const.) determined by $\omega_n(x) = 1$. From the sequence $\{\omega_n\}$ of positive solutions we can extract a subsequence which converges to a nonnegative solution ω uniformly on compact subsets of $\Omega \cup \partial\Omega$. Since $\omega(x) = 1$, $\omega | \partial\Omega = 0$, we have $\Omega \notin SO_{PN}$. Thus $d < \infty$ implies as before that $SO_{P\Phi} = \emptyset$.

That $SO_{P\Phi} \subset SO_{PB}$ follows from Lemma 3 and to complete the proof we show $SO_{PB} \subset SO_{P\Phi}$, if $d = \infty$. To this end suppose $\Omega \notin SO_{P\Phi}$, that is $\partial\Omega$ is regular and there exist solutions u, v on Ω such that $\Phi(|u|) \leqq v$ and u vanishes continuously on $\partial\Omega$. Set $w_n = \omega_n / c_n$ and extract a subsequence again denoted by $\{w_n\}$ which converges to a solution w uniformly on compact subsets of $\Omega \cup \partial\Omega$. We want to show that for the fixed $x \in \Omega \cap R_1$, $w(x) \neq 0$.

We let I be the positive bounded linear functional which assigns to each $f \in C(\partial(R_n \cap \Omega))$ the solution to the Dirichlet problem with boundary values f evaluated at x. By Jensen's inequality applied to the integral representation of I we have

$$\Phi(I(|u| w_n)/I(w_n)) \leqq (I(\Phi(|u|)w_n)/I(w_n)).$$

But $I(|u| w_n) = I(|u|) \geqq |u(x)|$ and $I(\Phi(|u|)w_n) \leqq I(vw_n) \leqq I(v) = v(x)$. Thus

$$\begin{aligned}\Phi(|u(x)|/w_n(x)) &\leqq v(x)/w_n(x) \\ &= (v(x)/|u(x)|)(|u(x)|/w_n(x)).\end{aligned}$$

Since $d = \infty$, the above implies that $\{|u(x)|/w_n(x)\}$ is bounded, i.e. $w_n(x) \to w(x) > 0$.

5. We now establish the

THEOREM. *If* $P \not\equiv 0$, *then* (ii) *is valid for* $i = 0$ *and* 1.

Indeed Lemma 3 gives dim $PB(R) > i$ implies that dim $P\Phi(R) > i$. It remains to show that dim $P\Phi(R) > i$ implies that dim $PB(R) > i$, $i = 0$, 1 for then a simple logical manipulation gives the assertion.

If dim $P\Phi(R) > 1$, then there exists a Φ-bounded solution v which changes sign. Indeed if v_1 and v_2 are linearly independent Φ-bounded solutions that do not change sign, then we assume they are of opposite sign. We fix $x \in R$ and $\lambda \in (0, 1)$ such that $\lambda v_1(x) + (1-\lambda)v_2(x) = 0$. By the convexity of Φ the solution $v = \lambda v_1 + (1-\lambda)v_2$ is Φ-bounded and must change sign by the Harnack principle. Set $\Omega_1 = \{v < 0\}$ and $\Omega_2 = \{v > 0\}$. In view of Theorem 4 and Theorem 1 we have dim $PB(R) > 1$.

To complete the proof we need only show that dim $PB(R) > 0$ follows from the assumption that dim $P\Phi(R) = 1$. The result of Naim cited above on Φ-bounded solutions being the difference of nonnegative ones allows us to assert the existence of v, a positive Φ-bounded solution on R. By Sard's theorem there is a $c > \inf v$ such that the hypersurface $v^{-1}(c)$ is smooth. Thus there is a bounded solution v' on a component Ω of the set $\{v > c\}$ such that $v \geqq v' \geqq 0$ and v' has continuous boundary value c on $\partial\Omega$. Since $\Phi(v - v') \leqq \Phi(v)$ we again apply Theorem 4 and Theorem 1 to conclude that dim $PB(R) > 0$.

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