ON THE COMPLETENESS OF HAMILTONIAN VECTOR FIELDS

WILLIAM B. GORDON

Abstract. Sufficient conditions are given for a hamiltonian vector field to be complete which involve bounds on the potential or its gradient.

1. A vector field \( \xi \) on a manifold \( M \) is said to be complete iff for every \( m \in M \) the maximal interval of existence \( (\omega_-, \omega_+) \) of every solution of

\[
\frac{dx}{dt} = \xi, \quad x(0) = m,
\]

is given by \( \omega_\pm = \pm \infty \), cf. [1]. Hence if \( \xi \) is of class \( C^1 \), so that solutions are unique, to say that \( \xi \) is complete means that \( \xi \) generates a 1-parameter group or (global) flow on \( M \). Let \( (t_-, t_+) \) be a bounded open neighborhood of 0 in which a solution \( x = x(t) \) to (1) is defined. It is well known that \( \omega_\pm = \pm \infty \) iff \( x(t) \) remains in a compact set as \( t \) varies over any such neighborhood, [2]. A device which assures this property is provided by the following simple lemma.

Lemma. Let \( \xi \) be a \( C^0 \) vector field on a manifold \( M \) of class \( C^1 \). Then \( \xi \) is complete if there exist a \( C^1 \) function \( E \), a proper \( C^0 \) function \( f \), and constants \( \alpha, \beta \) such that for all \( m \in M \)

(i) \( \| \xi E(m) \| \leq \alpha \| E(m) \| \),
(ii) \( |f(m)| \leq \beta |E(m)| \).

(Recall that \( f \) proper means \( f^{-1}(\text{compact}) = \text{compact} \).)

Proof. From basic definitions \( \xi E(m) = dE(x(t))/dt \|_{t=0} \). Hence from Gronwall's inequality, or otherwise, it follows that \( |E(x(t))| \leq |E(x(0))| e^{\alpha|t|}, \omega_- < t < \omega_+ \) so that \( |f(x(t))| \leq \beta |E(x(0))| e^{\alpha|t|} \). Since \( f \) is proper, this means that \( x(t) \) remains in a compact set as \( t \) varies over a bounded neighborhood of 0 (for which a solution is defined).

2. To apply this lemma to the case \( \xi = \) hamiltonian vector field, let \( (M, g) \) be a riemannian manifold of class \( C^1 \) with metric tensor \( g \), and let \( (q, p) \) denote local coordinates on \( T^*(M) \). Every \( C^1 \) "potential" \( V \) on \( M \) gives rise to a hamiltonian \( H = T + V = \) kinetic energy + potential \( = \frac{1}{2} \sum g^{ij} \dot{p}_i \dot{p}_j + V(q) \) whose corresponding hamiltonian vector...
field $\xi$ on $T^*(M)$ is given by $\xi = \sum \left\{ (\partial H/\partial p_i)\partial/\partial q_i - (\partial H/\partial q_i)\partial/\partial p_i \right\}$. We now prove the following

**Theorem.** Let $(M, g), \xi, V$ be as above. Then $\xi$ is complete if any of the following are true.

(i) $V$ is proper and bounded below, say $V \geq 0$.
(ii) $(M, g)$ is complete (in the riemannian sense) and $V \geq 0$.
(iii) $(M, g)$ is complete and $\| \nabla V \|$ is bounded.
(iv) $(M, g)$ is complete and $\| \nabla V \| \leq \text{constant} \cdot \| w \|$ where $m \rightarrow w(m)$ is an isometric embedding of $M$ into euclidean space.

**Proof.** For (i) apply the lemma with $f = E = H$. Since $V$ is a proper function on $M$, and $V \geq 0$, it follows that $H$ is proper on $T^*(M)$. But $\xi H = 0$, so that the hypotheses of the lemma are satisfied. For the remainder of the theorem, let $g \rightarrow w(g)$ be an isometric embedding of $M$ into euclidean space $\mathbb{R}^n$. (The existence of such embeddings is given by a theorem of Nash [3].) Since $(M, g)$ is complete as a riemannian manifold, $M$ is complete with respect to the metric induced by $g$. Therefore $M$ is a closed submanifold of $\mathbb{R}^n$. It follows that $r^2(m) = \| w(m) \|^2$ is a proper function on $M$. Also we have

$$| \xi r^2 | = 2 | \langle w, \sum p_i w_i \rangle | \leq 2 \| w \| \cdot \| \sum p_i w_i \| \leq 2 \| w \| \cdot (2T)^{1/2}$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$, $w_i = \partial w/\partial q_i$ and $w^i = \sum g^{ij} w_j$, so that $g_{ij} = \langle w_i, w_j \rangle$ and $g^{ij} = \langle w^i, w^j \rangle$. To prove (ii), set $E = H + r^2$. Then it is easy to show that $| \xi E/E | \leq 2$. To obtain a function $f$ satisfying the hypotheses of the lemma, set $f = r^2$ (or $f = E$). To prove (iii) and (iv) let $f = E = T + \frac{1}{2}r^2$. A direct calculation shows that $| \xi T | = | \sum g^{ij} p_i \partial V/\partial q_i | \leq (2T)^{1/2} \cdot \| \nabla V \|$ so that $| \xi E/E | \leq (2T)^{1/2} \| w \| \left( 1 + \| \nabla V \| / \| w \| \right) / (T + \frac{1}{2} \| w \| ^2)$. The proof of (iv) is now immediate. To prove (iii) we need only choose an embedding for which $\| w(m) \|$ is bounded below by a positive number.

Note that (i) has as a consequence the well-known fact that every hamiltonian vector field attached to (the cotangent bundle of) a compact manifold is complete, this being a generalization of the fact that every compact riemannian manifold is complete in the riemannian sense, (the case $V = 0$).

Finally, we remark that the lemma and the theorem can easily be extended to the nonautonomous case $\xi = \xi(m, t)$ by the usual device: one considers the vector field $\xi = \xi + \partial/\partial t$ on $M \times \mathbb{R}$ (in the lemma) and $T^*(M) \times \mathbb{R}$ (in the theorem) to obtain sufficient conditions for completeness. For example, parts (iii) and (iv) of the theorem remain true if the potential is time dependent; to see this replace the function $E$ in the argument by $E = E + t^2$. Then $E$ is a proper function
on $T^*(M) \times \mathbb{R}$ and $|\xi\dot{E}/\dot{E}| \leq$ constant. On the other hand, generalizations of parts (i) and (ii) apparently require boundedness conditions on $\partial V/\partial t$.

Bibliography


Naval Research Laboratory, Code 7840, Washington, D. C. 20390