OSCILLATION THEOREMS FOR NONLINEAR
SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Recently, Bobisud, Proc. Amer. Math. Soc. 23 (1969), 501–505, has discussed the oscillatory behavior of solutions of \( y'' + a(t)f(y) = 0 \). The purpose of this paper is to prove Bobisud's theorems under weaker assumptions. The proofs are quite different from his and simpler.

1. Introduction. Recently, Bobisud [2] has discussed the oscillatory behavior of solutions of

\[ y'' + a(t)f(y) = 0. \]

Let \( S \) denote the set of all solutions of (1) which exist on some positive half-line \([T_y, \infty)\), where \( T_y \) depends on the particular solution \( y \). A solution of (1) is said to be oscillatory if it has arbitrarily large zeros, and (1) is said to be oscillatory if every solution of (1) is oscillatory.

With \( f(y) = y^{2n+1} \) \((n \geq 1)\) and \( a > 0 \), Atkinson [1] proved that all solutions of (1) are oscillatory if and only if

\[ \int_x^\infty ta(t)\,dt = +\infty. \]

Many authors (cf. [3], [4]) have extended the results of Atkinson, and Bobisud [2] has proved oscillation theorems for (1) without the restriction \( a > 0 \).

The purpose of this paper is to show that Theorem 1 and Theorem 2 [2] remain valid without the assumption

\[ 0 < \lim_{t \to +\infty} \int_x^t a(s)\,ds \leq +\infty \]

and to prove Theorem 3 [2] under weaker conditions. The proofs of our theorems are quite different from Bobisud's and moreover very easy.

2. Oscillation theorems.

THEOREM 1. Let \( f \) be continuous and continuously differentiable on \( (-\infty, 0) \cup (0, \infty) \) with \( uf(u) > 0 \), \( f' \geq 0 \) there; let \( f \) also satisfy

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Finally, assume

\[ \int_{x}^{\infty} sa(s)ds = + \infty. \]

Then every \( y \in S \) is either oscillatory or tends monotonically to zero, as \( t \to \infty \).

**Proof.** Let \( y \in S \) be nonoscillatory and assume that \( y > 0 \), for some \( p > 0 \) on \( [p, \infty) \); a similar argument treats the case \( y < 0 \). Thus, from (1), we obtain

\[ ty''/f(y) = - ta(t). \]

By integration from \( p \) to \( t \) and integration by parts, we obtain

\[
\frac{ty'(t)}{f(y(t))} = \int_{p}^{t} \frac{y'(t)}{f(y(t))} dt - \int_{p}^{t} \frac{tf'(y(t))(y'(t))^2}{(f(y(t)))^2} dt + \frac{py'(p)}{f(y(p))}
- \int_{p}^{t} ta(t)dt.
\]

Since \( f'(y(t)) \geq 0 \), we find

\[ \frac{ty'(t)}{f(y(t))} \leq \int_{y(p)}^{y(t)} \frac{1}{f(s)} ds - \int_{p}^{t} sa(s)ds + \frac{py'(p)}{f(y(p))}. \]

The first integral in the right-hand side is bounded above, by the hypothesis (B). Thus, (3) implies that

\[ \frac{ty'(t)}{f(y(t))} \to - \infty, \quad \text{as } t \to \infty. \]

So, we obtain for some \( k > 0 \),

\[ \frac{ty'(t)}{f(y(t))} < - k \quad \text{for every } t \geq \exists q \geq p. \]

By integration from \( q \) to \( t \), we obtain for \( t \geq q \),

\[ \int_{y(q)}^{y(t)} \frac{1}{f(s)} ds \leq \ln\left( \frac{q}{t} \right)^k. \]

From (5), we conclude \( \lim_{t \to \infty} y(t) = 0 \). Thus, we obtain that \( y \in S \) is oscillatory or tends monotonically to zero as \( t \to \infty \).
Theorem 2. In addition to the hypotheses of Theorem 1, assume that for some $x > 0$,
\[ \lim_{t \to \infty} \int_{t}^{x} \frac{du}{f(u)} < \infty, \quad \lim_{t \to 0+} \int_{t}^{-x} \frac{du}{f(u)} < \infty. \]
Then every $\gamma \in S$ is oscillatory.

Proof. Proceeding as above, we have only to show that $y \in S$ does not tend monotonically to zero, as $t \to \infty$. This follows immediately from (5), since the left-hand side of (5) is bounded below by (C).

Theorem 3. In addition to the hypotheses of Theorem 1, assume that $f' \geq \gamma > 0$ on $(-\infty, 0) \cup (0, \infty)$, that $a(s)$ and $a_-(s)$ ($= \min(a(s), 0)$) is integrable over $(0, \infty)$, and that
\[ 0 \leq \lim_{t \to \infty} \int_{-x}^{t} a(s)ds + \frac{\gamma^2}{x} \leq + \infty. \]
Then every $\gamma \in S$ is oscillatory.

Proof. As in the proof of Theorem 2, we have only to show that $y \in S$ does not tend monotonically to zero, as $t \to \infty$. Assume then that $y > 0$, $y' < 0$ on $[p, \infty)$; a similar argument treats the case $y < 0$, $y' > 0$. From (4), we obtain, for some $\gamma > 0$,
\[ \frac{y'(t)}{f(y(t))} < - \frac{\gamma}{t}, \quad \text{for all } t \geq q \geq p. \]
Thus,
\[ \int_{q}^{t} f'(y(s)) \left[ \frac{y'(s)}{f(y(s))} \right]^2 ds > \int_{q}^{t} \frac{\gamma^2}{s^2} ds = \gamma^2 \left[ \frac{1}{q} - \frac{1}{t} \right]. \]
From (1), it follows (cf. [2, p. 502]) that
\[ \frac{y'(t)}{f(y(t))} - \frac{y'(q)}{f(y(q))} + \int_{q}^{t} f'(y(s)) \left[ \frac{y'(s)}{f(y(s))} \right]^2 ds + \int_{q}^{t} a(s)ds = 0. \]
By means of Theorem 3 and (7), we obtain from (8)
\[ \lim_{t \to \infty} \frac{y'(t)}{f(y(t))} < 0. \]
This leads to a contradiction as, in Bobisud's argument [2, p. 504], and the proof is complete.
Theorem 4. Let \( f \) be continuous and continuously differentiable on \((-\infty, 0) \cup (0, \infty)\) with \( uf(u) > 0, f' \geq 0 \) there; let \( f \) also satisfy

\[
\int_0^\infty du/f(u) < \infty, \quad \int_0^{-\infty} du/f(u) < \infty.
\]

Assume that \( a(t) \) is continuous and \( ta(t) \) is integrable. Then \( y \in S \) is oscillatory if and only if

\[
\int_{-\infty}^{\infty} ta(t)dt = +\infty.
\]

Proof. The necessity is included in Theorem 2. The sufficiency follows immediately from the proof of Theorem 1 [4, p. 113].

Theorem 5. In addition to the hypotheses of Theorem 1, assume that \( a(t) \) be eventually nonnegative. Then every \( y \in S \) is oscillatory.

Proof. Let \( y \in S \) be nonoscillatory. Then we may without loss of generality assume that \( y(t) > 0 \) for \( t \geq \Xi q \geq p \). From (1), we obtain \( y'' \leq 0 \). Hence \( y'(t) \geq 0 \) for \( t \geq q \), which contradicts (4).

Remarks. Macki and Wong [4] have discussed Theorem 4 in the case \( a(t) > 0 \). Theorem 5 shows that Theorem 2 [3] is valid for \( n = 1 \). Although Theorem 5 is included in Macki-Wong's Theorem [4], it is important that Theorem 5 is extended to arbitrary even order differential equations (cf. [3]).

References