MEASURES OF N-FOLD SYMMETRY FOR CONVEX SETS[1]

CHARLES K. CHUI AND MILTON N. PARNES

ABSTRACT. If a convex set S is 3-fold symmetric about a point $0\in S$, then any 3-star contained in S with vertex 0 is no smaller than any other parallel 3-star contained in S. In this paper, among other results, we establish the converse. Consequently, we find two measures of n-fold symmetry, one for $n=2$, 3, and the other for each $n\geqq 2$.

1. **Introduction.** A set S in the complex plane $\mathbf{C}$ is said to be n-fold symmetric about a point P if every rotation about P of $2\pi/n$ maps S onto itself. Let $\mathfrak{F}$ be the family of all compact convex sets in $\mathbf{C}$ with nonempty interior, that is, different from a line segment. Following Grünbaum [2], we call a real-valued function F a *similarity invariant measure of n-fold symmetry* for $\mathfrak{F}$, if

(i) $0\leqq F(K)\leqq 1$, $K\in\mathfrak{F}$;

(ii) $F(K)=1$ if and only if $K\in\mathfrak{F}$ has a center of n-fold symmetry;

(iii) $F(K)=F(T(K))$ for every $K\in\mathfrak{F}$ and every nonsingular similarity transformation T of $\mathbf{C}$ onto itself;

(iv) F is continuous on $\mathfrak{F}$.

There is a large literature on such measures in case $n=2$ (cf. [2]). In this paper we are going to introduce two definitions of measures for other natural numbers n. [See §3.] Our first definition, which holds for $n=2$, 3, comes from Theorem A. This theorem also gives a partial answer to a conjecture in the second author's dissertation [4]. Our second definition, which holds for all $n\geqq 2$, is based on Theorem B, a generalization of Hammer's result [3].

We shall use the same terminology as introduced in [5]. Let S be a convex set in $\mathbf{C}$, $0\in S$ and let T be an arbitrary n-star at 0 contained in S. That is, T is any n-star with vertex 0 and rays terminating on the boundary of S. Then S is said to have the *n-maximal property* at 0 if every n-star U, contained in S and parallel to T, is no larger than T: $|U|\leqq|T|$. The second author [4], [5] proved that *every bounded convex set S, n-fold symmetric about $0\in S$, has the n-*

Received by the editors October 10, 1969.

AMS 1969 *subject classifications*. Primary 5230.

Key words and phrases. N-fold symmetry, convex sets, similarity invariant measure, n-maximal property, n-supporting-line property, n-star.

[1] Some results of this paper are based on the second author's doctoral dissertation written under the direction of Professor W. Seidel.

maximal property at 0 *for every natural number* $n \geqq 2$. He also conjectures that *if a convex set* S *has the* p*-maximal property at a point* $0 \in S$, *where* p *is a prime, then* S *should be* p*-fold symmetric about* 0. It is easy to see that the condition p being a prime cannot be omitted. For if S has the n-maximal property at 0, it also has the mn-maximal property at 0 for every natural number m. In particular, every rectangle has the 4-maximal property at its center 0, although it is only 2-fold symmetric about 0. The above conjecture is contained in the following:

CONJECTURE. *Let* S *be a bounded convex set in* $\mathbf{C}$ *with the* n*-maximal property at* $0 \in S$, *and let* $n = p_1 \cdots p_k$, *where* $p_1, \cdots, p_k$ *are primes. Then* S *is* p_j*-fold symmetric about* 0 *for some* $j = 1, \cdots, k$.

The following theorem gives a partial answer to this conjecture.

THEOREM A. *Let* G *be a bounded convex set in* $\mathfrak{F}$ *with the* n*-maximal property at* $0 \in G$. *Then if* $n = 2, 3$, G *is* n*-fold symmetric about* 0, *and if* $n = 4$, G *is* 2*-fold symmetric about* 0.

The result for $n = 2$ is not new, and easily follows from a result of Hammer [**3**].

Let S be a bounded convex set in $\mathbf{C}$ containing the origin 0 and let $\rho(\theta) = \rho(\theta + 2\pi)$, $-\infty < \theta < \infty$, be the polar representation of the boundary of S. We say that S has the *n-supporting-line property with respect to* 0, if for every θ, there are n lines of support of S at the angles of $\theta, \theta + 2\pi/n, \cdots, \theta + 2(n-1)/n$, such that the angle between any two adjacent lines is $(n-2)\pi/n$. It is clear that if S is n-fold symmetric about 0, it has the n-supporting-line property with respect to 0. We will prove the converse, namely

THEOREM B. *Let* $n \geqq 2$ *be an arbitrary natural number and let* G *be a bounded convex set in* $\mathfrak{F}$ *having the* n*-supporting-line property with respect to an interior point* 0 *of* G. *Then* G *is* n*-fold symmetric about* 0.

2. **Proofs of Theorems A and B.** We first prove Theorem B. Let C be the boundary curve of G with polar representation $\rho = \rho(\theta)$, and let $\psi = \psi(\theta)$ denote the angle at (θ, ρ) measured from the radial line to the line of support in the counterclockwise direction. Since G is convex, C has continuous turning tangents at all but a countable number of points, and by elementary calculus, we know that at these points

$$\operatorname{ctn} \psi(\theta) = \rho'(\theta)/\rho(\theta).$$

The n-supporting-line property implies

$$\psi(\theta) = \psi(\theta + 2\pi/n)$$

for all θ. Hence, integrating ρ'/ρ and taking exponentials, we see that there exists a positive constant c such that

$$\rho(\theta) = c\rho(\theta + 2\pi/n) = \cdots = c^{n-1}\rho(\theta + 2(n-1)\pi/n) = c^n\rho(\theta).$$

Hence, $c=1$ and G is n-fold symmetric about 0.

We shall prove Theorem A for $n=3$. The proofs for $n=2$, 4 are similar and easier. We first list the following five geometric observations, the proofs of which are quite elementary.

(1) Let Δ be an equilateral triangle with center 0 and interior G. Let T be a 3-star contained in G with vertex 0, and let U be any 3-star contained in G and parallel to T. Then $|U| \leqq |T|$ and equality holds if and only if either the rays of U fall on different sides of Δ or $U=T$.

(2) Let T be a 3-star of finite length, and let r_1, r_2, r_3 be its rays. Let U be another 3-star parallel to T with rays $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{r}_3$ such that the vertex of U lies on r_3; r_1, r_2, $\tilde{r}_1$, $\tilde{r}_2$ terminate on a common straight line; and r_3, $\tilde{r}_3$ terminate at the same point. Then $|T| \leqq |U|$, and equality holds if and only if $T=U$.

(3) Let T be a 3-star with rays r_1, r_2, r_3 and let r_1, r_2 terminate on straight lines L_1, L_2 respectively, where either L_1 and L_2 are parallel or they intersect on the same side of the line passing through the tips of r_1, r_2 as r_3. Then it is possible to construct a 3-star U with rays $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{r}_3$ parallel to r_1, r_2, r_3 respectively, such that $\tilde{r}_j$ terminates on L_j, $j=1, 2$; r_3 and $\tilde{r}_3$ terminate at the same point; $r_3 \subset \tilde{r}_3$; and $|T| < |U|$.

(4) Let T be a 3-star contained in a triangle Δ such that the vertex of T lies in the interior of Δ and each ray of T terminates at an interior point of a different side of Δ. Construct the three equilateral triangles Δ_1, Δ_2, Δ_3 such that one side of each Δ_j lies on an extended side of Δ and the other two sides of Δ_j pass through the other two tips of the rays of T. Then at least one of Δ_1, Δ_2, Δ_3 contains T in its interior.

(5) Let Δ be a triangle which is not equilateral, and let T be a 3-star contained in Δ such that the vertex of T lies in the interior of Δ and each ray of T terminates at an interior point of a different side of Δ. Then there is a 3-star U contained in Δ and parallel to T such that the vertex of U is arbitrarily close to that of T and $|U| > |T|$.

The proofs of (1) through (4) are quite straightforward, while the proof of (5) follows from (1) through (4). As a corollary of these observations, we obtain

THEOREM 2.1. *Let G be the interior of a convex polygon P, T a 3-star with vertex in G and contained in G such that $|T| \geqq |U|$ where U is any 3-star contained in G and parallel to T. Then either at least one ray of T terminates at a vertex of P or the three rays of T terminate on the interior of three different sides of P, which, when extended, form an equilateral triangle that contains T in its interior.*

PROOF. Suppose that the three rays of T terminate on the interior of the sides of P. By (2) these rays terminate on different sides, and by (3) these three sides, when extended, form a triangle which contains G in its interior. By (5) this triangle is indeed equilateral.

We remark that for the special case of convex polygons Theorem A is already proved. Indeed, from the above proof we see that 3-maximal property implies 3-supporting-line property which in turn, by Theorem B, implies 3-fold symmetry. To include a larger collection of convex sets, we need the following

LEMMA 2.1. *Let G be a convex domain with boundary C. Let T be a 3-star with vertex $0 \in G$ and rays terminating on C at points where C has continuous turning tangents, such that T is no smaller than any parallel 3-star contained in G. Then the lines of support L_1, L_2, L_3 of G at the tips a_1, a_2, a_3, respectively, of T form an equilateral triangle.*

PROOF. The lines of support form one of the configurations as described in our above five observations. For instance, assume that they form a triangle, which is not equilateral, such that T terminates on the interior of its three different sides as in observation (4). Let $L_1 = L_1'$ be the line of support on which an equilateral triangle as described in the conclusion of (4) can be constructed. Let L_2' and L_3' be drawn through a_2 and a_3 respectively to form this equilateral triangle. One can find a 3-star T^* contained in G, parallel to T and with vertex $0^* \in G$ arbitrarily close to 0 so that the rays of T^* intersect L_2' and L_3' in G or on C. [Cf. Fig. a or Fig. b.] Using the fact that a tangent line to a curve is a much closer approximation to the curve than any secant line, one can easily see that if 0^* is suitably chosen $|T^*| > |T|$. This is a contradiction. The proofs for the other configurations are similar.

We can now complete the proof of Theorem A ($n=3$). Let C be the boundary curve of G. Then C has continuous turning tangents at all but a countable number of points. Hence, by using the right-hand derivatives and Lemma 2.1, G has the 3-supporting-line property with respect to 0, and is, therefore, 3-fold symmetric about 0 by Theorem B.

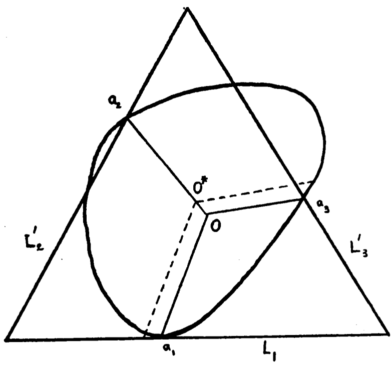

FIGURE a

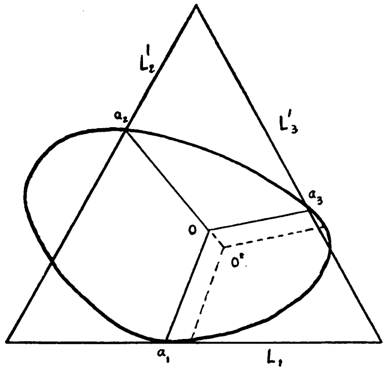

FIGURE b

3. **Measures of symmetry.** Let $G\in\mathfrak{F}$, P a point in G and let P_θ, $0\leqq\theta<2\pi/n$, denote the n-star contained in G, with vertex P, and having a ray with an angle of inclination θ measured positively from the real axis. Let

$$M_n(G;P) = \inf_{Q,\theta} \frac{|P_\theta|}{|Q_\theta|},$$

where Q runs over G and $0\leqq\theta<2\pi/n$. We define the function M_n on $\mathfrak{F}$ by

$$M_n(G) = \sup_{P\in G} M_n(G;P), \qquad G\in\mathfrak{F}.$$

THEOREM 3.1. *For $n=2$, 3, M_n is a similarity invariant measure of n-fold symmetry for $\mathfrak{F}$.*

PROOF. It is clear that M_n satisfies (i) in §1. Since each $G\in\mathfrak{F}$ has nonempty interior, the useful n-stars Q_θ in the definition of $M_n(G;P)$ have lengths bounded away from zero; hence, M_n is continuous. Finally, for $n=2$, 3, we see that, using Theorem A, $M_n(G)=1$, $G\in\mathfrak{F}$, if and only if G is n-fold symmetric.

Note that in defining M_n, we have required that the sets G to have nonempty interior, that is, to be different from a line segment. In the latter case, it is not clear how to interpret $|P_\theta|/|Q_\theta|$, when both $|P_\theta|$ and $|Q_\theta|$ are zero for all, except one, values of θ, $0\leqq\theta<2\pi/n$. Indeed, in this case, continuity for M_3 breaks down: If we approximate a line segment L by rectangles, the limit would be $\frac{1}{2}$; however, if we approximate L by isosceles triangles, it would be $2/5$.

We now define our second measure based on Theorem B. Let

$G \in \mathfrak{F}$, P a point in G and P_θ as defined above, $0 \leqq \theta < 2\pi/n$. Let Δ be a regular n-gon such that G is contained in the closure of its interior and such that if Δ' is another regular n-gon which lies in the interior of Δ, then G does not lie in the closure of the interior of Δ'. Such a Δ will be called admissible. Let Q be the center of Δ. (If $n=2$, Δ is a pair of parallel lines and Q is equidistant from these two lines.) Let Q_θ be the n-star contained in the interior of Δ with vertex Q and parallel to P_θ. We define

$$N_n(G; P) = \inf_{\Delta} \sup_{\theta} \frac{|P_\theta|}{|Q_\theta|},$$

where $0 \leqq \theta < 2\pi/n$ and Δ runs over all admissible regular n-gons. We now define the function N_n on $\mathfrak{F}$ by

$$N_n(G) = \sup_{P \in G} N_n(G; P), \qquad G \in \mathfrak{F}.$$

THEOREM 3.2. *For each natural number $n \geqq 2$, N_n is a similarity invariant measure of n-fold symmetry for $\mathfrak{F}$.*

PROOF. For $G \in \mathfrak{F}$, it is obvious that $0 \leqq N_n(G) \leqq 1$. If G has a positive diameter d, each admissible Δ has diameter no less than d. Hence, the corresponding Q_θ has length bounded away from d. Continuity of N_n follows. By Theorem B, it is not difficult to see that $N_n(G) = 1$, $G \in \mathfrak{F}$, if and only if G is n-fold symmetric.

Note that different from M_n ($n=3$, say), the continuity of N_n even holds for a line segment L. In fact, approximating L by sets in $\mathfrak{F}$, we should define $N_2(L) = 1$ and $N_3(L) = \frac{1}{2}$. Next, we find a lower bound of $M_3(G)$, $G \in \mathfrak{F}$, larger than zero. We need the following lemma (cf. [1]).

LEMMA 3.1. *Every convex set of diameter d is contained in a circle of diameter no greater than $2d/\sqrt{3}$.*

THEOREM 3.3. *For each $G \in \mathfrak{F}$, $1/2\sqrt{3} \leqq N_3(G) \leqq 1$.*

PROOF. For an admissible Δ, let Q_θ be the corresponding 3-star parallel to P_θ. Using Lemma 3.1 and elementary geometry, we see that

$$\sup_{\theta} \frac{|P_\theta|}{|Q_\theta|} \geqq \frac{1}{2\sqrt{3}}$$

for every admissible Δ. Hence, the result follows.

REFERENCES

1. H. G. Eggleston, *Convexity*, Cambridge Tracts in Math. and Math. Phys., no. 47, Cambridge Univ. Press, New York, 1958, p. 111. MR **23** #A2123.

2. B. Grünbaum, *Measures of symmetry for convex sets*, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., Providence, R. I., 1963, pp. 233–270. MR **27** #6187.

3. P. C. Hammer, *Diameters of convex bodies*, Proc. Amer. Math. Soc. **5** (1954), 304–306. MR **15**, 819.

4. M. Parnes, *Symmetrization and conformal mapping*, Ph.D. thesis, Wayne State Univ., Detroit, Mich., 1968.

5. ———, *A distortion theorem for doubly connected regions*, Proc. Amer. Math. Soc. **26** (1970), 85–91.

STATE UNIVERSITY OF NEW YORK AT BUFFALO, AMHERST, NEW YORK 14226