NO INFINITE DIMENSIONAL P SPACE ADMITS A MARKUSCHEVICH BASIS

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ABSTRACT. THEOREM. *Let X be a Banach space. If X is a Grothendieck space and X admits a Markuschevich basis then X is reflexive.* This theorem is used to prove the conjecture of J. A. Dyer [1] stated in the title.

Recall that a Banach space X is a Grothendieck space if every weak* convergent sequence in X^* is weakly convergent. X is a P space if X is complemented in every Banach space which contains it as a subspace.[1] Since a complemented subspace of a Grothendieck space is a Grothendieck space and since every P space can be embedded in the Grothendieck space $m(T)$ for a suitable set T, every P space is a Grothendieck space. Infinite dimensional P spaces are nonreflexive, so Dyer's conjecture is a consequence of our theorem.

PROOF OF THE THEOREM. Suppose that $\{x_i, f_i\}_{i\in I}$ is a Markuschevich basis for the Grothendieck space X; i.e., $\{x_i, f_i\}_{i\in I}$ is a biorthogonal collection in (X, X^*) such that $\{x_i\}_{i\in I}$ is fundamental in X and $\{f_i\}_{i\in I}$ is total over X. Let Y be the norm closure in X^* of the linear span of $\{f_i\}_{i\in I}$ and let B be the closed unit ball of Y.

To show that X is reflexive it is sufficient to show that Y is reflexive. (Indeed, Y is total over X so that Y is weak* dense in X^*. If B is weakly compact,[2] then B is weak* compact, so that it follows from the Krein-Smulian theorem that Y is weak* closed and hence $Y=X^*$.) By Eberlein's theorem, we need to show only that B is weakly sequentially compact.

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in B. Since each y_n is the norm limit of a sequence from the linear span of $\{f_i\}_{i\in I}$, it follows that for each n, the set $A_n=\{i\in I: y_n(x_i)\neq 0\}$ is countable and thus $\bigcup_{n=1}^{\infty} A_n$ is countable. A standard diagonalization argument shows that there is an increasing sequence $\{P(n)\}_{n=1}^{\infty}$ of positive integers such that $\lim_{n\to\infty} y_{P(n)}(x_i)$ exists for each $i\in I$. Since $\{y_n\}_{n=1}^{\infty}$ is equicontinuous

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[1] For the basic facts concerning P spaces see [3]. The most interesting nonreflexive Grothendieck spaces are discussed in [2].

[2] Since the weak topology on Y by Y^* is the relativisation to Y of the weak topology on X^* by X^{**}, there is no ambiguity in discussing the weak topology on Y.

on X and $\{x_i\}_{i\in I}$ is fundamental in X, $\lim_{n\to\infty} y_{P(n)}(x)$ exists for each $x\in X$. That is, $\{y_{P(n)}\}_{n=1}^{\infty}$ is weak* convergent to, say, y in X^*. Since X is a Grothendieck space, $\{y_{P(n)}\}_{n=1}^{\infty}$ is weakly convergent to y. Finally, y is in Y (and hence in B) because the weak and norm closures in X^* of the linear span of $\{f_i\}_{i\in I}$ are the same. Thus B is weakly sequentially compact and the proof is complete.

References

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