NO INFINITE DIMENSIONAL P SPACE ADMITS A MARKUSCHEVICH BASIS

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Abstract. Theorem. Let X be a Banach space. If X is a Grothendieck space and X admits a Markuschevich basis then X is reflexive. This theorem is used to prove the conjecture of J. A. Dyer [1] stated in the title.

Recall that a Banach space X is a Grothendieck space if every weak* convergent sequence in X* is weakly convergent. X is a P space if X is complemented in every Banach space which contains it as a subspace. Since a complemented subspace of a Grothendieck space is a Grothendieck space and since every P space can be embedded in the Grothendieck space m(T) for a suitable set T, every P space is a Grothendieck space. Infinite dimensional P spaces are nonreflexive, so Dyer's conjecture is a consequence of our theorem.

Proof of the Theorem. Suppose that \( \{x_i, f_i\}_{i \in I} \) is a Markuschevich basis for the Grothendieck space X; i.e., \( \{x_i, f_i\}_{i \in I} \) is a biorthogonal collection in \((X, X^*)\) such that \( \{x_i\}_{i \in I} \) is fundamental in X and \( \{f_i\}_{i \in I} \) is total over X. Let Y be the norm closure in X* of \( \{f_i\}_{i \in I} \) and let B be the closed unit ball of Y.

To show that X is reflexive it is sufficient to show that Y is reflexive. (Indeed, Y is total over X so that Y is weak* dense in X*. If B is weakly compact, then B is weak* compact, so that it follows from the Krein-Smulian theorem that Y is weak* closed and hence Y = X*.) By Eberlein's theorem, we need to show only that B is weakly sequentially compact.

Let \( \{y_n\}_{n=1}^\infty \) be a sequence in B. Since each \( y_n \) is the norm limit of a sequence from the linear span of \( \{f_i\}_{i \in I} \), it follows that for each n, the set \( A_n = \{i \in I : y_n(x_i) \neq 0\} \) is countable and thus \( \bigcup_{n=1}^\infty A_n \) is countable. A standard diagonalization argument shows that there is an increasing sequence \( \{P(n)\}_{n=1}^\infty \) of positive integers such that \( \lim_{n \to \infty} y_{P(n)}(x_i) \) exists for each \( i \in I \). Since \( \{y_n\}_{n=1}^\infty \) is equicontinuous

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1 For the basic facts concerning P spaces see [3]. The most interesting nonreflexive Grothendieck spaces are discussed in [2].

2 Since the weak topology on Y by Y* is the relativisation to Y of the weak topology on X* by X**, there is no ambiguity in discussing the weak topology on Y.
on $X$ and $\{x_i\}_{i \in I}$ is fundamental in $X$, \( \lim_{n \to \infty} y_{P(n)}(x) \) exists for each $x \in X$. That is, $\{y_{P(n)}\}_{n=1}^\infty$ is weak* convergent to, say, $y$ in $X^*$. Since $X$ is a Grothendieck space, $\{y_{P(n)}\}_{n=1}^\infty$ is weakly convergent to $y$. Finally, $y$ is in $Y$ (and hence in $B$) because the weak and norm closures in $X^*$ of the linear span of $\{f_i\}_{i \in I}$ are the same. Thus $B$ is weakly sequentially compact and the proof is complete.

References