CARDINALS m SUCH THAT $2m = m$

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ABSTRACT. In this paper we compare characterizations of cardinals m satisfying $2 \cdot m = m$ with certain characterizations of Dedekind infinite cadinals. It is also shown that a strengthening of $\forall m$ (m an infinite cardinal $\Rightarrow 2 \cdot m = m$) implies the axiom of choice.

1. In [2] several results of the form $\neg(\mathrm{ZF} \vdash \mathrm{S} \rightarrow \mathrm{AC})$, where S is a sentence implied by AC in ZF and $\neg(\mathrm{ZF} \vdash \mathrm{S})$, are mentioned. If S is the sentence

$$\forall x\,(x \textit{ infinite} \Rightarrow 2 \cdot |x| = |x|), \tag{1}$$

where $\cdot$ is cardinal multiplication, it is unknown whether or not $\mathrm{ZF} \vdash \mathrm{S} \rightarrow \mathrm{AC}$.

The purpose of this paper is to show that a certain strengthening of (1) implies AC in ZF and to compare cardinals m which satisfy $2 \cdot m = m$ with Dedekind infinite cardinals.

2. We will be working in Zermelo-Fraenkel set theory (ZF) without the axiom of choice (AC). Statements of theorems and proofs will be given in informal mathematical language but are translatable into the formal language of ZF.

We assume that the cardinal of a set x (denoted $|x|$) has been defined so that

(i) $|x| = |y|$ if and only if there is a one-to-one function from x onto y;

(ii) the cardinal number of any finite set is an integer; and

(iii) $||x|| = |x|$.

$\aleph_0$ denotes the cardinal number of the set of nonnegative integers.

(1) is somewhat ambiguous since two definitions of "x is infinite" which agree under the assumption of AC may not agree without AC.

DEFINITION 2.1. x is *infinite* if for all integers n, there is no 1-1 function from x into n.

DEFINITION 2.2. x is *Dedekind infinite* if $|x| \geqq \aleph_0$.

This ambiguity is resolved by the following theorem.

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THEOREM 2.3.

$$\forall x\,(\aleph_0 \leqq |x| \Rightarrow 2\cdot|x| = |x|) \Rightarrow \forall x\,(x \textit{ infinite} \Rightarrow \aleph_0 \leqq |x|).$$

(Hence the weaker form of (1) *implies the stronger form.)*

PROOF. Assume $\forall x(\aleph_0 \leqq |x| \Rightarrow 2\cdot|x| = |x|)$ and let κ be any infinite cardinal (in the sense of 2.1). Then $\kappa+\aleph_0 \geqq \aleph_0$, so by our assumption $(\kappa+\aleph_0)+(\kappa+\aleph_0)=\kappa+\aleph_0$. Therefore $\kappa+\kappa+\aleph_0=\kappa+\aleph_0$. Let f be a 1-1 function from $\kappa\times\{1\}\cup\aleph_0\times\{2\}$ onto $\kappa\times 2\cup\aleph_0\times\{2\}$ and let

$$x_0 = f[\kappa\times\{1\}]\cap\kappa\times\{0\}, \qquad x_1 = f[\kappa\times\{1\}]\cap\kappa\times\{1\},$$
$$n_0 = f[\aleph_0\times\{2\}]\cap\kappa\times\{0\}, \qquad n_1 = f[\aleph_0\times\{2\}]\cap\kappa\times\{1\}.$$

If n_0 is infinite then $|n_0| = \aleph_0$, hence $\kappa \geqq \aleph_0$ and we are done, similarly if n_1 is infinite. So assume n_0 and n_1 are finite. Then since $|x_0|+|n_0|=\kappa$, x_0 is infinite and similarly for x_1. Hence

$$(|x_0|+|n_0|)+1 \leqq |x_0|+|x_1| \leqq \kappa = |x_0|+|n_0|$$

so that $(|x_0|+|n_0|)+1=|x_0|+|n_0|$, i.e. $\kappa+1=\kappa$. Hence κ is Dedekind infinite [3, p. 53].

3. The strengthening of (1) mentioned in §1 is the following:

(2) *For every set x there is a function f with domain x such that for every infinite $y\in x$, $f(y)$ is a* 1-1 *function from $2\times y$ onto y.*

THEOREM 3.1. (2)$\Rightarrow$AC.[2]

PROOF. Assume (2) and let z be any set. We will construct a choice function for z. Let

$$x = \{(w\times\aleph_0)\cup\{\varnothing\} : w\in z \quad\text{and}\quad w\neq\varnothing\}.$$

For $w\in z$ let $w^* = (w\times\aleph_0)\cup\{\varnothing\}$. Let f be the function on x given by (2). Define $g: z-\{\varnothing\}\to\bigcup z$ as follows:

$$g(w) = \pi_1(f(w^*)(\langle 1,\varnothing\rangle)) \quad\text{if } \varnothing\in f(w^*)[\{0\}\times w^*],$$
$$g(w) = \pi_1(f(w^*)(\langle 0,\varnothing\rangle)) \quad\text{if } \varnothing\in f(w^*)[\{1\}\times w^*]$$

where $\pi_1(\langle x,y\rangle)=x$ for any ordered pair $\langle x,y\rangle$.

Then $g(w)\in w$ for each $w\in z$, $w\neq\varnothing$.

The idea of this proof can be used to establish that another statement, apparently weaker than (2), also implies AC, namely:

(2′) *For every set x there is a function f with domain x such that for every infinite element y of x, $f(y)$ is an unordered pair of* 1-1 *functions*

[2] Mr. Gordon Monro of Bristol University independently discovered the stronger version of this theorem which is mentioned below. His proof is essentially the same as the one given here.

both of which have domain y and whose ranges constitute a 2-partition of y.

(2′) is apparently weaker than (2) because $f(y)$ is an *unordered* pair.

4. A cardinal m is Dedekind infinite $\Leftrightarrow m+\aleph_0=m \Leftrightarrow \exists\, \kappa$ (κ a cardinal and $\kappa+\aleph_0=m$). There is a similar characterization of cardinals m satisfying $2\cdot m=m$.

THEOREM 4.1. *For all cardinals* m, $2\cdot m=m \Leftrightarrow \aleph_0\cdot m=m \Leftrightarrow \exists\, \kappa$ *(*κ *a cardinal and* $\aleph_0\cdot\kappa=m$*).*

PROOF. Suppose $2\cdot m=m$. Then there are 1-1 functions f and h with domain m such that $\mathrm{Ra}(f)\cap\mathrm{Ra}(h)=\varnothing$ and $\mathrm{Ra}(f)\cup\mathrm{Ra}(h)=m$.

For $\langle n, u\rangle\in\aleph_0\times m$ let $g(\langle n, u\rangle)=h^{(n)}(f(u))$. ($h^{(n)}$ is h composed with itself n times.) Then g maps $\aleph_0\times m$ into m and g is 1-1, for suppose $g(\langle n, u\rangle)=g(\langle j, w\rangle)$, then $h^{(n)}(f(u))=h^{(j)}(f(w))$. If $n>j$ we have $h^{(n-j)}(f(u))=f(w)$, since h is 1-1. But this is impossible since h and f have disjoint ranges. Similarly if $j>n$ we get a contradiction. So $n=j$ and $f(u)=f(w)$. But f is 1-1 so $u=w$ and $\langle n, u\rangle=\langle j, w\rangle$.

This gives $\aleph_0\cdot m\leqq m$, hence $\aleph_0\cdot m=m$.

Now suppose there is a cardinal κ such that $\aleph_0\cdot\kappa=m$. We define $g:\aleph_0\times\kappa\to 2\times m$ as follows:

$$g(\langle 2n, u\rangle) = \langle 0, f(\langle n, u\rangle)\rangle, \qquad g(\langle 2n+1, u\rangle) = \langle 1, f(\langle n, u\rangle)\rangle,$$

where f is a 1-1 map from $\aleph_0\times\kappa$ onto m. g is clearly 1-1 and onto so $m=\aleph_0\cdot\kappa=2\cdot m$.

We now have $2m=m\Rightarrow\aleph_0\cdot m=m$ and $\exists\,\kappa$ (κ a cardinal and $\kappa\cdot\aleph_0=m)\Rightarrow 2\cdot m=m$. The other implications follow easily.

Another characterization of Dedekind infinite cardinals is the following:

m is Dedekind infinite $\Leftrightarrow$ there is a function f mapping m 1-1, into m with $\mathrm{Ra}(f)\neq m$.

which has this analogue:

THEOREM 4.2. *For all cardinals* m, $2m=m \Leftrightarrow$ *there is a function* f *mapping* m 1-1, *into* m *with* $\mathrm{Ra}(f)\neq m$ *and satisfying*

$$\forall\, x\in m\ \ \exists\, n<\aleph_0(f^{(-n)}(x)\notin\mathrm{Ra}(f)).$$

PROOF. Suppose first that there is a 1-1 function f satisfying the conditions of the theorem. Let $A=\mathrm{Ra}(f)$.

For $x\in A$ define $g(x)$ to be the unique n such that $f^{(-n)}(x)\notin A$. (Hence $f^{(-n+1)}(x)\in A$.) Let $h(x)=f^{(-g(x))}(x)$. Finally let $k(x)$

$=\langle g(x), h(x)\rangle$. The domain of k is A and clearly k maps A 1-1, onto $\aleph_0 \times (m-A)$. Hence $m=|A|=\aleph_0 \cdot |m-A|$. So by 4.1, $2\cdot m=m$.

Now suppose $2\cdot m=m$. As in Theorem 4.1 let f and h be two 1-1 functions with domain m such that $\mathrm{Ra}(f)\cap\mathrm{Ra}(h)=\varnothing$ and $\mathrm{Ra}(f)\cup\mathrm{Ra}(h)=m$. Let $A=\{x\in\mathrm{Ra}(f): \exists\, n<\aleph_0(f^{(-n)}(x)\notin\mathrm{Ra}(f))\}$, and let $m'=f^{-1}[A]$. Now $|A|=m$, since $|A|\leqq m$ and $f[\mathrm{Ra}(h)]\subseteq A$. Also $A\subseteq m'$, for suppose $x\in A$, then $x=f^{-1}(y)$ for some $y\in\mathrm{Ra}(f)$. If $f^{(-n)}(x)\notin\mathrm{Ra}(f)$, then $f^{(-n-1)}(y)\notin\mathrm{Ra}(f)$ so that $y\in A$, hence $x\in f^{-1}[A]=m'$.

So $f|m'$ maps m' 1-1 and onto $A\subset m'$. Now for any $x\in A$, $\exists\, n<\aleph_0$ such that $f^{(-n)}(x)\notin\mathrm{Ra}(f)$, hence $f^{(-n)}(x)\notin A$. It also follows from this that $A\neq m'$. Now since $|m'|=m$ the desired result follows.

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