SEVERAL THEOREMS ON BOUNDEDNESS AND EQUICONTINUITY[1]

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ABSTRACT. This paper presents several results concerning equicontinuity of a pointwise-bounded family of linear transformations on a Banach space. The first is the following generalization of the Banach-Steinhaus Theorem: Let $\{T_\alpha | \alpha \in A\}$ be a pointwise-bounded family of linear transformations from a Banach space X to a normed linear space Y, and assume that, for each $\alpha \in A$, T_α is continuous on a closed subspace S_α of X. Then $\exists\ \alpha_1, \cdots, \alpha_n \in A$ such that the family is equicontinuous on $\bigcap_{k=1}^n S_{\alpha_k}$. The second theorem deals with a pointwise-bounded family of linear transformations from a Banach space X to a normed linear space with a continuous bilinear mapping into another normed linear space. The others deal with homomorphisms of Banach algebras.

The purpose of this paper is to present several theorems illustrating a method by which equicontinuity can be deduced from pointwise-boundedness, for families of linear transformations whose domain is a Banach space.

The first theorem simultaneously generalizes the classical Banach-Steinhaus Theorem and a Uniform Boundedness Theorem of Pták [4].

THEOREM 1. *Let $\mathfrak{F} = \{T_\alpha | \alpha \in A\}$ be a pointwise-bounded family of linear transformations from a Banach space X to a normed space Y. Let S_α be a closed subspace of X such that $T_\alpha | S_\alpha$ is continuous, $\forall\ \alpha \in A$. Then $\exists\ \alpha_1, \cdots, \alpha_n \in A$ such that $\mathfrak{F}$ is equicontinuous on $\bigcap_{k=1}^n S_{\alpha_k}$.*

PROOF. For $x \in X$ let $M(x) = \sup\{\|T_\alpha x\| \,|\, \alpha \in A\}$, and let $r(\alpha)$ denote the norm of T_α restricted to S_α. Assume the theorem false. Then, given any finite set of pairs $(S_1, T_1), \cdots, (S_p, T_p)$ with $T_k \in \mathfrak{F}$ nonzero and continuous on the closed subspace S_k, $1 \leqq k \leqq p$, and given any constant $M > 0$, there is a $T \in \mathfrak{F}$ and $x \in \bigcap_{k=1}^p S_k$ with $\|x\| \leqq 1$, $\|Tx\| > M$, and $\|T_k x\| \leqq 1$ for $1 \leqq k \leqq p$.

Since $\mathfrak{F}$ is not equicontinuous on any S_α, choose $T_1 \in \mathfrak{F}$ and $x_1 \in X$ with $\|x_1\| \leqq 1$, $\|T_1 x_1\| > 4$. Inductively, choose sequences

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$\{x_n \mid n=1, 2, \cdots\} \subseteq X$, $\{T_n \mid n=1, 2, \cdots\} \subseteq \mathfrak{F}$ with $\|x_n\| \leqq 1$, $x_n \in \bigcap_{k=1}^{n-1} S_k$ for $n \geqq 2$, where S_k is the closed subspace on which T_k is continuous, and $\|T_k x_n\| \leqq 1$ for $1 \leqq k \leqq n-1$, $\|T_n x_n\| > 2^n(n+1+\sum_{j=1}^{n-1} 2^{-j} M(x_j))$. Let $x = \sum_{k=1}^{\infty} 2^{-k} x_k$, by completeness, $x \in X$. For $n \geqq 2$, $\|T_n x\| = \|2^{-n} T_n x_n + \sum_{k=1}^{n-1} 2^{-k} T_n x_k + T_n(\sum_{k=n+1}^{\infty} 2^{-k} x_k)\|$. But $k \geqq n+1 \Rightarrow x_k \in S_n$, and since S_n is closed and $T_n \mid S_n$ is continuous, $T_n(\sum_{k=n+1}^{\infty} 2^{-k} x_k) = \sum_{k=n+1}^{\infty} 2^{-k} T_n x_k$, and so

$$\|T_n x\| \geqq 2^{-n}\|T_n x_n\| - \sum_{k=1}^{n-1} 2^{-k}\|T_n x_k\| - \sum_{k=n+1}^{\infty} 2^{-k}\|T_n x_k\|$$

$$\geqq 2^{-n}\left[2^n\left(n+1+\sum_{k=1}^{n-1} 2^{-k} M(x_k)\right)\right] - \sum_{k=1}^{n-1} 2^{-k} M(x_k) - \sum_{k=n+1}^{\infty} 2^{-k},$$

since $1 \leqq k \leqq n-1 \Rightarrow \|T_n x_k\| \leqq M(x_k)$ and $k \geqq n+1 \Rightarrow \|T_n x_k\| \leqq 1$. So, $\|T_n x\| \geqq n+1-\sum_{k=n+1}^{\infty} 2^{-k} > n$, contradicting the pointwise-boundedness of $\mathfrak{F}$. Q.E.D.

The Banach-Steinhaus Theorem occurs when $S_\alpha = X$, $\forall \alpha \in A$, and the Pták Theorem when each T_α has closed null-space $N(T_\alpha)$, and $S_\alpha = N(T_\alpha)$, $\forall \alpha \in A$.

The second theorem concerns mappings into a space with a continuous bilinear form, and is proved in a similar fashion.

DEFINITION. A relation $x \perp y$ on a normed linear space is called an orthogonality relation if $x \perp y$ and $x \perp z \Rightarrow x \perp y+z$, $x \perp y$ and α a scalar $\Rightarrow x \perp \alpha y$, and $x \perp x_n$ for $n=1, 2, \cdots$; $x_n \to x_0 \Rightarrow x \perp x_0$. Given an orthogonality relation $\perp$ on a normed linear space X, and given also $x \in X$, we define $S^\perp(x)$ to be the closed linear subspace $\{y \in X \mid x \perp y\}$.

THEOREM 2. *Let X be a Banach space, Y, Z normed linear spaces and let $\mathfrak{F} = \{T_\alpha \mid \alpha \in A\}$ be a pointwise-bounded family of linear transformations from X to Y. Let $\perp$ be an orthogonality relation on X, and let $q: Y \times Y \to Z$ be a continuous bilinear (or conjugate-linear) form. Assume that for each $(\alpha, x) \in A \times X$, the maps $y \to q(T_\alpha x, T_\alpha y)$ is continuous on $S^\perp(x)$. Then $\exists\, x_1, \cdots, x_n \in X$ such that $1 \leqq i \leqq j \leqq n \Rightarrow x_i \perp x_j$ and $\exists\, M > 0$ such that $\alpha \in A$, $x \in \bigcap_{k=1}^{n} S^\perp(x_k) \Rightarrow \|q(T_\alpha x, T_\alpha x)\| \leqq M\|x\|\,\|T_\alpha x\|$.*

PROOF. Assume the theorem is false. We assert that, given a finite set $x_1, \cdots, x_n \in X$ with $x_i \perp x_j$ for $1 \leqq i \leqq j \leqq n$, $M > 0$, and a finite set $T_1, \cdots, T_n \in \mathfrak{F}$, there is an $x \in X$ and $T \in \mathfrak{F}$ with $x_i \perp x$ $(1 \leqq i \leqq n)$, $\|x\| \leqq 1$, $\|q(T_k x_k, T_k x)\| \leqq \|T_k x_k\|$ $(1 \leqq k \leqq n)$, and $\|q(Tx, Tx)\| > M\|Tx\|$. For $y \in X$, define $M(y) = \sup\{\|T_\alpha y\| \mid \alpha \in A\}$, and for $(x, \alpha) \in X \times A$, define

$$r(x, \alpha) = \inf\{\delta \mid y \in S^{\perp}(x) \Rightarrow \|q(T_\alpha x, T_\alpha y)\| \leq \delta\|T_\alpha x\| \, \|y\|\}.$$

We can assume that, for $1 \leq k \leq n$, $r(x_k, k) \neq 0$, for otherwise the requirement that $\|q(T_k x_k, T_k x)\| \leq \|T_k x_k\|$ is trivially satisfied. By assumption, we can find $y \in X$ with $x_k \perp y$ $(1 \leq k \leq n)$ and $T \in \mathfrak{F}$ with $\|q(Ty, Ty)\| > \gamma M\|y\| \, \|Ty\|$, where $\gamma = \max(r(x_1, 1), \cdots, r(x_n, n), 1)$. Let $x = (\gamma\|y\|)^{-1}y$, then clearly $\|x\| \leq 1$. We also have

$$\begin{aligned} \|q(Tx, Tx)\| &= (\gamma\|y\|)^{-2}\|q(Ty, Ty)\| > (\gamma\|y\|)^{-2}\gamma M\|Ty\| \, \|y\| \\ &= M(\gamma\|y\|)^{-1}\|Ty\| = M\|Tx\| \end{aligned}$$

and, for $1 \leq k \leq n$,

$$\begin{aligned} \|q(T_k x_k, T_k x)\| &= (\gamma\|y\|)^{-1}\|q(T_k x_k, T_k y)\| \\ &\leq (\gamma^{-1}r(x_k, k))\|y\|^{-1}\|T_k x_k\| \, \|y\| \leq \|T_k x_k\|. \end{aligned}$$

Assuming the theorem false, choose $x_1 \in X$, $T_1 \in \mathfrak{F}$ with $\|x_1\| \leq 1$, $\|q(T_1 x_1, T_1 x_1)\| > 4\|T_1 x_1\|$. Having chosen $x_1, \cdots, x_{n-1}$ and $T_1, \cdots, T_{n-1}$, choose $x_n \in X$, $T_n \in \mathfrak{F}$ with $\|x_n\| \leq 1$, $x_i \perp x_n$, $\|q(T_i x_i, T_i x_n)\| < \|T_i x_i\|$ for $1 \leq i \leq n-1$ and

$$\|q(T_n x_n, T_n x_n)\| > 2^n\left(n + 1 + \sum_{k=1}^{n-1} 2^{-k}M(x_k)\right)\|T_n x_n\|.$$

Let $x = \sum_{k=1}^{\infty} 2^{-k}x_k$, by completeness, $x \in X$. We can assume without loss of generality that $y, z \in Y \Rightarrow \|q(y, z)\| \leq \|y\| \, \|z\|$. Then for each integer n,

$$\begin{aligned} \|T_n x\| \, \|T_n x_n\| \geq \|q(T_n x_n, T_n x)\| &= \left\| q\left(T_n x_n, T_n\left(\sum_{k=1}^{\infty} 2^{-k}x_k\right)\right)\right\| \\ &= \left\| 2^{-n}q(T_n x_n, T_n x_n) + \sum_{j=1}^{n-1} 2^{-j}q(T_n x_n, T_n x_j) \right. \\ &\qquad \left. + \, q\left(T_n x_n, T_n\left(\sum_{k=n+1}^{\infty} 2^{-k}x_k\right)\right)\right\|. \end{aligned}$$

Since $S^{\perp}(X_n)$ is a closed subspace, $k \geq n+1 \Rightarrow x_k \in S^{\perp}(x_n)$, and $y \to q(T_n x_n, T_n y)$ is continuous on $S^{\perp}(x_n)$, so we see that

$$q\left(T_n x_n, T_n\left(\sum_{k=n+1}^{\infty} 2^{-k}x_k\right)\right) = \sum_{k=n+1}^{\infty} 2^{-k}q(T_n x_n, T_n x_k).$$

Now $1 \leq k \leq n-1 \Rightarrow \|q(T_n x_n, T_n x_k)\| \leq \|T_n x_n\| \, \|T_n x_k\| \leq M(x_k)\|T_n x_n\|$ and $k \geq n+1 \Rightarrow \|q(T_n x_n, T_n x_k)\| \leq \|T_n x_n\|$. So

$$\|T_nx\|\,\|T_nx_n\| \geqq 2^{-n}\|q(T_nx_n, T_nx_n)\| - \sum_{k=1}^{n-1} 2^{-k}\|q(T_nx_n, T_nx_k)\|$$

$$- \sum_{k=n+1}^{\infty} 2^{-k}\|q(T_nx_n, T_nx_k)\|$$

$$> 2^{-n}\left[2^n\left(n+1+\sum_{k=1}^{n-1}2^{-k}M(x_k)\right)\right]\|T_nx_n\|$$

$$- \sum_{k=1}^{n-1} 2^{-k}\|T_nx_n\|M(x_k) - \sum_{k=n+1}^{\infty} 2^{-k}\|T_nx_n\|$$

$$= \left(n+1-\sum_{k=n+1}^{\infty}2^{-k}\right)\|T_nx_n\| > n\|T_nx_n\|;$$

since $T_nx_n \neq 0$ we have $\|T_nx\| > n$, contradicting the pointwise-boundedness of $\mathfrak{F}$. Q.E.D.

Although the Banach-Steinhaus Theorem and its uses are well known, the second theorem is a bit more obscure. The following is an example of its utility. Let H, K be Hilbert spaces, $T: H \to K$ a linear map such that $(x, y) = 0$ in $H \Rightarrow |(Tx, Ty)| \leqq C\|x\|\,\|y\|$. Then T is continuous. Theorem 2 yields the existence of $x_1, \cdots, x_n$ such that $(x, x_k) = 0$ for $1 \leqq k \leqq n \Rightarrow |(Tx, Tx)| \leqq M\|Tx\|\,\|x\|$, and so $\|Tx\|^2 \leqq M\|Tx\|\,\|x\| \Rightarrow \|Tx\| \leqq M\|x\|$. Letting B denote the closed subspace spanned by $x_1, \cdots, x_n$, we have $H = B \oplus B^\perp$, and $T|B^\perp$ is continuous. But B is finite dimensional, and so $T|B$ is continuous, completing the proof.

These techniques can also be applied to homomorphisms of Banach algebras. One can modify the Main Boundedness Theorem of Badé and Curtis to obtain

THEOREM 3. *Let $\{T_\alpha | \alpha \in A\}$ be a pointwise-bounded family of homomorphisms of a Banach algebra X into a Banach algebra Y. Let $\{x_n | n = 1, 2, \cdots\}$, $\{y_n | n = 1, 2, \cdots\}$ be sequences from X such that $x_ny_m = 0$ for $n \neq m$. Then*

$$\sup\{\|T_\alpha(x_ny_n)\|/\|x_n\|\,\|y_n\| \mid \alpha \in A, n = 1, 2, \cdots\} < \infty.$$

Here we have used the formulation of the Main Boundedness Theorem given in [1]. Techniques similar to those used in Theorem 1 will show that the theorems of [2] and [5] which relate to continuity of a homomorphism also hold for equicontinuity of a pointwise-bounded family of homomorphisms. For example, Theorem II.1 of [5] can be changed to show that every pointwise-bounded family of

Banach algebra homomorphisms of a von Neumann algebra is equicontinuous on a dense subalgebra.

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