ON REPRESENTATIONS OF SELFMAPPINGS

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ABSTRACT. It is shown in this note that every "mild" selfmapping $f:X\to X$ of a compact Hausdorff space X into itself can be represented by the product $(Y,g)\times(Z,h)$ of two selfmappings g and h, where g is a contraction ($\bigcap_1^\infty g^n(Y)=$singleton) and h is a homeomorphism of Z onto itself. Endowing the set of all selfmappings X^X with the compact-open topology, the qualifier "mild" means that the closure of the family $\{f^n \mid n\geqq 1\}\subset X^X$ is compact. In case X is metrizable, some results of M. Edelstein and J. de Groot are used to linearize (X, f) in the separable Hilbert space.

1. **Introduction and notation.** Let X be a Hausdorff topological space and $f:X\to X$ a continuous mapping of X into itself. In this case we call the pair (X,f) briefly a *selfmap*. If (Y, g) is another selfmap, (Y, g) is said to be a *representation* of (X,f) or to *represent* (X,f) iff there exists a topological embedding $\tau:X\to Y$ of X into Y such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau} & Y \\ f\downarrow & & \downarrow g \\ X & \xrightarrow{\tau} & Y \end{array}$$

is commutative, which means that the action of g restricted to $\tau(X)$ is identical with $\tau f\tau^{-1}$.

Under the Cartesian product of two selfmaps (X,f) and (Y, g) we understand the pair $(X\times Y, f\times g)$ where $f\times g:X\times Y\to X\times Y$ is defined by

$$(f\times g)(x, y) = (f(x), g(y)) \qquad \text{for all } x\in X \quad \text{and} \quad y\in Y.$$

A selfmap $(L, 1)$ is called *linear* iff L is a linear topological space and $1:L\to L$ a continuous linear operator on it. If $(L, 1)$ represents (X,f) we say (X,f) is *linearly represented* by $(L, 1)$.

If (X,f) is a selfmap and R an equivalence relation on X such that f preserves R ($xRy\Rightarrow f(x)Rf(y)$) then there is a uniquely determined selfmap $f_R:X/R\to X/R$ such that the diagram

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$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ p\downarrow & & \downarrow p \\ X/R & \xrightarrow{f_R} & X/R \end{array}$$

commutes where p is the natural projection. We call $(X/R, f_R)$ the quotient selfmap induced by R.

Let X be a topological space, we denote by X^X the set of all continuous selfmappings $f:X\to X$. Introducing in X^X the compact-open topology, X^X becomes a topological semigroup under functional composition. If $f\in X^X$ we denote by $\Gamma(f)$ the closure in X^X of the family $\{f^n | n\geqq 1\}$. Thus $\Gamma(f)$ is the minimal closed subsemigroup of X^X containing f. We say a selfmap (X,f) is *mild* iff $\Gamma(f)$ is compact, it is a *squeezing selfmap* iff the interesction $\bigcap_1^\infty f^n(X)$ of all iterated images $f^n(X)$ is a singleton: $\bigcap_1^\infty f^n(X) = \{a\}$ for some $a\in X$. Finally we say (X,f) is a homeomorphism if $f:X\to X$ maps X onto itself homeomorphically.

The main purpose of this note is to show that any mild selfmap on a compact Hausdorff space can be represented by the Cartesian product of a squeezing selfmap and a homeomorphism. The proof of this result depends heavily on some results of A. D. Wallace [1] and [2] and shows that the concept of a squeezing selfmap is in a certain sense complementary to that of a homeomorphism.

Finally using certain results of J. de Groot [3] and [4] and M. Edelstein [5] on linearization of selfmaps some especially simple linear representation of mild selfmaps will be found in case X is compact and metrizable.

2. **Selfmaps on compact spaces.** Let (X, f) be a selfmap where X is compact Hausdorff and $A\subset X$ a nonempty closed invariant subset of X $(f(A)\subset A)$. Let R_A be the equivalence relation $R_A = A\times A\cup\{(x, x) | x\in X\}$, thus the quotient space X/R_A is the space X with A identified to a point. X/R_A is obviously again compact Hausdorff and the relation R_A is preserved under f. Denoting this space by X/A and by f_A the corresponding induced map we have the quotient selfmap $(X/A, f_A)$.

Proposition 2.1. *Let (X, f) be a selfmap, X compact Hausdorff, $A\subset X$ a closed nonempty invariant subset of X such that there is a continuous retraction $r:X\to A$ of X onto A which commutes with f. Then (X,f) is represented by $(X/a, f_A)\times(a, f|A)$, where $f|A$ denotes the restriction of f to A.*

PROOF. Denoting by $p:X\to X/A$ the natural projection we define the embedding

$$\tau:X \to X/A \times A \quad \text{by} \quad \tau(x) = (p(x), r(x)) \quad \text{for all } x \in X.$$

We first observe that τ is continuous and one-to-one, in fact if $x_1 \neq x_2$ and x_1 or x_2 or both $\overline{\in} A$ then $p(x_1) \neq p(x_2)$, and if $x_1 \neq x_2$, $x_1, x_2 \in A$ then $r(x_1) = x_1$, $r(x_2) = x_2$ so in both cases we have $\tau(x_1) \neq \tau(x_2)$. X being compact τ is a topological embedding and commutativity of the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\tau} & X/A \times A \\ f \downarrow & & \downarrow f_A \times f|A \\ X & \xrightarrow{\tau} & X/A \times A \end{array}$$

follows readily from the fact that r and f commute:

$$(f_A, f|A)\tau(x) = (f_A(p(X)), f(r(x))),$$
$$\tau(f(x)) = (p(f(x)), r(f(x))).$$

We apply this lemma to the special case when $A = \bigcap_1^\infty f^n(X)$. Due to the compactness of X, A is nonempty closed and f maps A onto itself. Much more can be said assuming f to be mild. Then due to the "Swelling Lemma" of A. D. Wallace [1] and [2] we have these results:

(i) The restriction $f|A$ is a homeomorphism of A onto itself.

(ii) There exists the unique idempotent $e \in \Gamma(f)$ which is a retraction of X onto A and since $\Gamma(f)$ is commutative this retraction $e:X\to A$ commutes with f.

Observing that in this case the quotient selfmap $(X/A, f_A)$ is squeezing since $\bigcap_1^\infty f_A^n[X/A] = \{p(A)\}$ and using the Proposition 2.1 we are now in position to prove our main theorem:

THEOREM 2.1. *Let (X, f) be a mild selfmap where X is compact Hausdorff. Then (X,f) has a representation by the product $(Y, g)\times(Z, h)$ where (Y, g) is squeezing, (Z, h) a homeomorphism and both Y, Z compact Hausdorff.*

PROOF. Set $Y = X/A$ with $A = \bigcap_1^\infty f^n(X)$, $g = f_A$, $Z = A$ and $h = f|A$. The following simple example illustrates this situation: Let X be $[-1, 1]$ and $f:X\to X$ defined by $f(x) = \frac{1}{2}x$ for $x \in [0, 1]$ and $f(x) = x$ for $x \in [-1, 0]$. Since f is nonexpanding, the family $\{f^n | n \geq 1\}$ is equicontinuous and therefore f is mild according to the Arzela-Ascoli Theorem. In this case we have $Y = [0, 1]$, $Z = [-1, 0]$, $g(x) = \frac{1}{2}x$ on Y and $h(x) = x$ on Z.

3. Linear representation of selfmaps. We now apply our results to the case when X is compact and metrizable. In this case due to some recent results of M. Edelstein [5] each squeezing selfmap can be linearly represented in l_2 in the following way:

THEOREM 3.1 (M. EDELSTEIN). *If X is compact metrizable and f a squeezing selfmap of X then to each $c\in(0, 1)$ there exists a topological embedding $\tau: X\to l_2$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\tau} & l_2 \\ f\downarrow & & \downarrow c\cdot P \\ X & \xrightarrow{\tau} & l_2 \end{array}$$

commutes where P is the continuous linear operator on l_2 defined by:

$$P(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots) \qquad \textit{for all } (x_1, x_2, x_3, \cdots) \in l_2.$$

According to some results of J. de Groot [3], each compact topological group acting continuously on a metrizable space X can be linearized in a suitable Hilbert space by a group of unitary transformations. In cases when X is compact, the theorem of J. de Groot can be stated as follows:

THEOREM 3.2 (J. DE GROOT). *Let G be a compact topological group acting continuously on the compact metrizable space X. Then there exists a topological embedding $\tau: X\to l_2$ and a one-to-one homomorphism $i: G \to U$ (U stands for the group of all unitary transformations of l_2 onto itself) such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\tau} & l_2 \\ g\downarrow & & \downarrow i(g) \\ X & \xrightarrow{\tau} & l_2 \end{array}$$

commutes for all $g\in G$.

Combining these two results we can easily prove our final theorem:

THEOREM 3.3. *Let (X, f) be a mild selfmap where X is compact and metrizable. Then to each $c\in(0, 1)$ there exists a representation of (X, f) by the product $(l_2, cP)\times(l_2, u)$ where P is defined in Theorem 3.1 and u is a unitary operator on l_2.*

PROOF OF THE THEOREM. Theorem 2.1 says that (X, f) is represented by $(X/A, f_A)\times(A, f|A)$ where $A=\bigcap_1^\infty f^n(X)$. Observing that

X/A is again metrizable and f_A squeezing we apply Theorem 3.1 to the first and Theorem 3.2 to the second factor of the above Cartesian product. Choosing $c \in (0, 1)$ arbitrarily Theorem 3.1 asserts the existence of a topological embedding $\tau_1: X/A \to l_2$ such that the diagram:

$$\begin{array}{ccc} X/A & \xrightarrow{\tau_1} & l_2 \\ f_A \downarrow & & \downarrow c \cdot P \\ X/A & \xrightarrow{\tau_1} & l_2 \end{array}$$

is commutative.

Observing that $f|A$ generates a compact group of homeomorphisms of A onto itself, Theorem 3.2 furnishes us with a topological embedding $\tau_2: A \to l_2$ and a unitary operator $u \in U$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tau_2} & l_2 \\ f|A \downarrow & & \downarrow u \\ A & \xrightarrow{\tau_2} & l_2 \end{array}$$

is commutative.

Combining these two results with Theorem 2.1 we get the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\tau} & X/A \times A & \xrightarrow{\tau_1 \times \tau_2} & l_2 \times l_2 \\ f \downarrow & & \downarrow f_A \times f|A & & \downarrow c \cdot P \times u \\ X & \rightarrow & X/A \times A & \xrightarrow{\tau_1 \times \tau_2} & l_2 \times l_2 \end{array}$$

proving our theorem.

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