A BARRELED SPACE WITHOUT A BASIS

N. J. KALTON

ABSTRACT. An example is given of a separable, barrelled, nuclear, bornological Ptak space which has no Schauder basis.

A sequence \( \{x_n\} \) in a locally convex space \( E \) is a basis if each \( x \) in \( E \) can be expressed uniquely in the form \( x = \sum a_n x_n \); Singer [5] has given an example of a separable locally convex space which does not possess a basis. In this note I shall give another example which has the additional properties of being barrelled, bornological, nuclear and a Ptak space; this also partially answers a question raised in [2].

It seems desirable to introduce another form of separability in locally convex spaces: \( E \) will be called \( \omega \)-separable if it possesses a subspace \( G \) of countable dimension and such that every member of \( E \) is the limit of a sequence in \( G \). Thus if \( E \) has a basis, \( E \) is \( \omega \)-separable, and if \( E \) is \( \omega \)-separable then \( E \) is separable. Let \( \mathfrak{N} \) denote the cardinal of the continuum and let \( \text{card}(X) \) denote the cardinal of any set \( X \). If \( E \) is \( \omega \)-separable then \( \text{card}(E) \) is less than the cardinal of the set of all sequences in \( G \); as \( \text{card}(G) = \mathfrak{N} \), \( \text{card}(E) = \mathfrak{N} \).

Let \( K \) be the field of real numbers or of complex numbers.

**Theorem.** \( K^\mathfrak{N} \) is barrelled, bornological, nuclear and a Ptak space; it is separable but not \( \omega \)-separable, and so does not possess a basis.

**Proof.** \( K^\mathfrak{N} \) has a weak topology and is complete; hence \( K^\mathfrak{N} \) is a Ptak space (see [4, p. 162]). By the Mackey-Ulam theorem ([3, §28.8]) it is bornological as \( \mathfrak{N} \) is not strongly inaccessible; a complete bornological space is barrelled. The product of nuclear spaces is nuclear [4, p. 102] and so \( K^\mathfrak{N} \) is nuclear. Finally, \( K^\mathfrak{N} \) is separable by Theorem 7.2, p. 175 of [1] but \( \text{card}(K^\mathfrak{N}) = 2^\mathfrak{N} > \mathfrak{N} \), and so \( K^\mathfrak{N} \) cannot be \( \omega \)-separable.

The question which naturally arises is: does every \( \omega \)-separable locally convex space possess a basis? Singer's example, the weak*-dual of the Banach space \( m \) of all bounded sequences, is not \( \omega \)-separable; this follows from the results of [5] or from the fact that \( m^* \) contains a copy of the Stone-Čech compactification of the integers, and so has cardinality \( 2^\mathfrak{N} \).

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References


Lehigh University, Bethlehem, Pennsylvania 18015