A BARRELLED SPACE WITHOUT A BASIS

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ABSTRACT. An example is given of a separable, barrelled, nuclear, bornological Ptak space which has no Schauder basis.

A sequence $\{x_n\}$ in a locally convex space E is a basis if each x in E can be expressed uniquely in the form $x=\sum_{i=1}^{\infty} a_i x_i$; Singer [5] has given an example of a separable locally convex space which does not possess a basis. In this note I shall give another example which has the additional properties of being barrelled, bornological, nuclear and a Ptak space; this also partially answers a question raised in [2].

It seems desirable to introduce another form of separability in locally convex spaces: E will be called ω-separable if it possesses a subspace G of countable dimension and such that every member of E is the limit of a sequence in G. Thus if E has a basis, E is ω-separable, and if E is ω-separable then E is separable. Let $\aleph$ denote the cardinal of the continuum and let card(X) denote the cardinal of any set X. If E is ω-separable then card(E) is less than the cardinal of the set of all sequences in G; as card$(G)=\aleph$, card$(E)=\aleph$.

Let K be the field of real numbers or of complex numbers.

THEOREM. *$K^{\aleph}$ is barrelled, bornological, nuclear and a Ptak space; it is separable but not ω-separable, and so does not possess a basis.*

PROOF. $K^{\aleph}$ has a weak topology and is complete; hence $K^{\aleph}$ is a Ptak space (see [4, p. 162]). By the Mackey-Ulam theorem ([3, §28.8]) it is bornological as $\aleph$ is not strongly inaccessible; a complete bornological space is barrelled. The product of nuclear spaces is nuclear [4, p. 102] and so $K^{\aleph}$ is nuclear. Finally, $K^{\aleph}$ is separable by Theorem 7.2, p. 175 of [1] but card$(K^{\aleph})=2^{\aleph}>\aleph$, and so $K^{\aleph}$ cannot be ω-separable.

The question which naturally arises is: does every ω-separable locally convex space possess a basis? Singer's example, the weak*-dual of the Banach space m of all bounded sequences, is not ω-separable;[1] this follows from the results of [5] or from the fact that m^* contains a copy of the Stone-Čech compactification of the integers, and so has cardinality $2^{\aleph}$.

Received by the editors February 6, 1970.

AMS 1969 *subject classifications*. Primary 4601.

Key words and phrases. Locally convex spaces, basis, separable, barrelled, nuclear.

[1] I am grateful to the referee for pointing this out.

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