AN APPROXIMATION THEOREM FOR $\bar{\partial}$-CLOSED FORMS OF TYPE $(n, n-1)$

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ABSTRACT. Let D be a bounded open set in $\mathbf{C}^n$ with smooth boundary. Then every closed form of type $(n, n-1)$ which is C^∞ on $\bar{D}$ can be approximated uniformly on $\bar{D}$ by $(n, n-1)$ forms which are closed in a neighborhood of $\bar{D}$. If $\mathbf{C}^n - D$ is connected these forms can be chosen to be closed in $\mathbf{C}^n$. This is applied to prove that a continuous function on the connected boundary of a bounded domain in $\mathbf{C}^n$ admits a holomorphic extension to the interior if and only if it is a weak solution of the tangential Cauchy-Riemann equations.

1. Statement of results. If D is an open set in $\mathbf{C}^n$ we denote by $E^{p,q}(D)$ the space of C^∞ complex differential forms on D of type (p, q) and by $E^{p,q}(\bar{D})$ the subspace of forms which have C^∞ extensions to a neighborhood of the closure, $\bar{D}$, of D. We give $E^{p,q}(D)$ the C^∞ topology (of uniform convergence on compact subsets of D of each coefficient and all its derivatives). We give $E^{p,q}(\bar{D})$ the topology of uniform convergence on $\bar{D}$ of each coefficient and all its derivatives. The main result of this note is the following theorem:

THEOREM 1. *If D is a bounded open subset of $\mathbf{C}^n$ with C' boundary, then every f in $E^{n,n-1}(\bar{D})$ such that $\bar{\partial} f = 0$ in D can be approximated in $E^{n,n-1}(\bar{D})$ by $(n, n-1)$ forms which satisfy $\bar{\partial} q = 0$ in a neighborhood of $\bar{D}$. If, further, the complement of $\bar{D}$ is connected, then these forms can be chosen to satisfy $\bar{\partial} g = 0$ in all of $\mathbf{C}^n$.*

As an application we obtain the following improvement on the results of [6]:

THEOREM 2. *If D is a bounded open subset of $\mathbf{C}^n$ with smooth connected boundary Γ, then a continuous function f on Γ has an analytic extension to D if and only if f is a weak solution of the tangential Cauchy-Riemann equations on Γ.*

Theorem 2 removes the hypothesis used in [6] that the Levi form of the domain D have one positive eigenvalue at each point of Γ.

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We remark that when $n=1$ Theorem 1 gives a special case of a well-known theorem of Mergelyan. Indeed, our proof of Theorem 1 was suggested in part by F. Browder's proofs [1] of approximation theorems of the Walsh-Mergelyan type for elliptic partial differential operators.

2. **Proof of Theorem 1.** We begin with the following lemma:

LEMMA. *The dual space of $E^{p,q}(\overline{D})$ can be identified with the space of (p, q)-forms whose coefficients are distributions with support in $\overline{D}$.*

PROOF. It suffices to consider the case (0, 0)-forms, i.e., C^∞ functions, since $E^{p,q}(\overline{D})$ is isomorphic to the direct sum of copies of $C^\infty(\overline{D})$. If T is a continuous linear functional on $C^\infty(\overline{D})$ define T on $C^\infty(\mathbf{C}^n)$ by $T(f)=T(f|\overline{D})$. If $f_n \in C^\infty(\mathbf{C}^n)$ and $f_n \to 0$ in the C^∞ topology on $\mathbf{C}^n$, then $f_n|\overline{D}\to 0$ in $C^\infty(\overline{D})$, hence $T(f_n)\to 0$. Hence the extension of T to $C^\infty(\mathbf{C}^n)$ is continuous, i.e., is a distribution with compact support, and clearly this support lies in $\overline{D}$. Conversely, if T is a distribution with support in $\overline{D}$ and $f\in C^\infty(\overline{D})$ we extend f to $f_1 \in C^\infty(\mathbf{C}^n)$ and define $T(f)$ to be $T(f_1)$. By Theorem XXXIII of Chapter III of [5], $T(f_1)$ depends only on f. By Theorem XXXIV of the same source this process defines a continuous linear functional on $C^\infty(\overline{D})$.

To prove the first statement of Theorem 1 it suffices, in view of this lemma and the Hahn-Banach theorem, to show that if T is a distribution $(n, n-1)$-form with support in $\overline{D}$ such that $Tf=0$ for all $f\in E^{n,n-1}(\overline{D})$ such that $\bar\partial f=0$ on a neighborhood of $\overline{D}$, then $Tf=0$ for all $f\in E^{n,n-1}(\overline{D})$ such that $\bar\partial f=0$ in D. For this we choose a sequence of open sets D_j such that $D_{j+1}\subset D_j$ and $\bigcap D_j=\overline{D}$. Recalling that the dual space of $C^\infty(D_j)$ is the space of distributions with compact support in D_j we regard T as an element of the dual space of $E^{n,n-1}(D_j)$ for each j.

The operator $\bar\partial: E^{n,n-1}(D_j)\to E^{n,n}(D_j)$ is continuous for each j. We claim that its range is all of $E^{n,n}(D_j)$. Indeed, if $g\in E^{n,n}(D_j)$, then

$$g = G dz_1 \wedge \cdots \wedge dz_n \wedge d\bar z_1 \wedge \cdots \wedge d\bar z_n$$

where $G\in C^\infty(D_j)$. Choose $H\in C^\infty(D_j)$ such that $\Delta H=-4G$ where Δ is the Laplacian on $\mathbf{R}^{2n}$. If

$$f = \sum_{i=1}^{n} (-1)^{n+i}(\partial H/\partial z_i) dz_1 \wedge \cdots (d\bar z_i)^\wedge \cdots \wedge d\bar z_n$$

where $(d\bar z_i)^\wedge$ indicates the absence of $d\bar z_i$, then $\bar\partial f=g$. Since $E^{n,n-1}(D_j)$ and $E^{n,n}(D_j)$ are Fréchet spaces, T annihilates the null space of $\bar\partial$, and the range of $\bar\partial$ is closed, it follows from a theorem of Dieudonné and

Schwartz [2] that T is in the range of the adjoint $\bar\partial^*$ of $\bar\partial$. Thus, for each j there exists S_j in the dual of $E^{n,n}(D_j)$ such that $\bar\partial^* S_j = T$.

Now, if $g \in E^{n,n}(D_1)$, then

$$S_j(g) = S_j(\bar\partial f) = \bar\partial^* S_j(f) = T(f).$$

Thus, considered as a subset of the dual space of $E^{n,n}(D_1)$, the set $\{S_j\}$ is weak*-bounded, and therefore has a weak*-cluster point by Theorem 18.8 (iii) of [4]. Since the support of S_j lies in D_j and since $D_j \searrow \bar D$ it follows that this cluster point, S, has support in $\bar D$. Also, since $S_j(\bar\partial f) = T(f)$ for all j and all $f \in E^{n,n-1}(D_1)$ we have $S(\bar\partial f) = T(f)$ so $\bar\partial^* S = T$.

Because D has a smooth boundary we can find open sets $U_1, \cdots, U_k$ and vectors $v_1, \cdots, v_k$ in $\mathbf{C}^n$ such that

(i) $\bar D \subset \bigcup_{i=1}^k U_i$;

(ii) if $p \in U_i \cap \bar D$, then $p + tv_i \in D$ for $0 < t < 1$.

Let $\{\phi_i\}$ be a C^∞ partition of unity subordinate to the covering $\{U_i\}$, and write $S = \sum S_i$ where $S_i = \phi_i S$. Let $f \in E^{n,n-1}(\bar D)$ be such that $\bar\partial f = 0$ in D. Then $T(f) = (\bar\partial^* S)(f) = \sum (\bar\partial^* S_i)(f)$. Let $f_{m,i}(p) = f(p + m^{-1} v_i)$ for $p \in U_i \cap \bar D$ and $m = 1, 2, \cdots$. It follows that $f_{m,i} \in E^{n,n-1}((U_i \cap \bar D)^-)$ and $f_{m,i} \to f \mid (U_i \cap \bar D)$ in $E^{n,n-1}((U_i \cap \bar D)^-)$ as $m \to \infty$. Moreover, $(\bar\partial f_{m,i})(p) = (\partial f)(p + m^{-1} v_i) = 0$ in a neighborhood of $(U_i \cap \bar D)^-$. Thus

$$\begin{aligned}(\bar\partial^* S_i)(f) &= (\bar\partial^* S_i)(f \mid U_i \cap \bar D) = \lim_{m\to\infty} (\bar\partial^* S_i)(f_{m,i}) \\ &= \lim_{m\to\infty} S_i(\bar\partial f_{m,i}) = 0,\end{aligned}$$

so that $T(f) = (\bar\partial^* S)(f) = \sum (\bar\partial^* S_i)(f) = 0$.

For the second part of the theorem suppose T is a distribution $(n, n-1)$-form with support in $\bar D$ such that $T(f) = 0$ for all $f \in E^{n,n-1}(\bar D)$ which have extensions to $\mathbf{C}^n$ satisfying $\bar\partial f = 0$ there. By the arguments used to prove the first statement in the theorem $T = \bar\partial^* S$ for some S in the dual space of $E^{n,n}(\mathbf{C}^n)$. But, by definition, if σ is the coefficient of S, the coefficients of $\bar\partial^* S$ are $(-1)^{n+i}(\partial\sigma/\partial\bar z_i)$, $1 \leqq i \leqq n$. Since T has support in $\bar D$, $\partial\sigma/\partial\bar z_i = 0$ in $\mathbf{C}^n - \bar D$ for each i. Now σ is a distribution with compact support. If we choose a C^∞ function ϕ with compact support in $\mathbf{C}^n$ such that $\phi \geqq 0$ and $\int \phi = 1$ and let $\phi_j(p) = j^n \phi(jp)$, then $\sigma_j = \sigma * \phi_j$ is a C^∞ function with compact support such that $\sigma_j(f) \to \sigma(f)$ for all $f \in C^\infty(\mathbf{C}^n)$ and such that $\partial\sigma_j/\partial\bar z_i = (\partial\sigma/\partial\bar z_i) * \phi_j$. Thus $\partial\sigma_j/\partial\bar z_i = 0$ in $\mathbf{C}^n - \bar D$ for each i, i.e., σ_j is holomorphic in $\mathbf{C}^n - \bar D$. But $\mathbf{C}^n - \bar D$ is connected and σ_j has compact support. Hence $\sigma_j = 0$ for each j, so $\sigma = 0$ in $\mathbf{C}^n - \bar D$. Thus S has support in $\overline{D}$. Now we can proceed as before to conclude that $T(f)=0$ if $f\in E^{n,n-1}(\overline{D})$ and $\bar{\partial}f=0$ in D.

3. **Proof of Theorem 2.** We recall from [6] that a continuous function f on Γ is a *weak solution of the tangential Cauchy-Riemann equations on* Γ if $\int_\Gamma f\bar{\partial}\mu=0$ for all $C^\infty(n, n-2)$-forms μ. We also recall the main theorem of [6]: *A continuous function f on Γ has an analytic extension to D if and only if $\int_\Gamma f\omega=0$ for all $\omega\in E^{n,n-1}(\overline{D})$ such that $\bar{\partial}\omega=0$ in D.*

Since Γ is connected, $\overline{D}$ has connected complement. Thus each $\omega\in E^{n,n-1}(\overline{D})$ such that $\bar{\partial}\omega=0$ in D can be approximated uniformly on $\overline{D}$ by ω_j satisfying $\bar{\partial}\omega_j=0$ in $\boldsymbol{C}^n$. By a well-known property of the $\bar{\partial}$-operator on $\boldsymbol{C}^n$ [3, Theorem 2.7.8], we have $\omega_j=\bar{\partial}\mu_j$ for $\mu_j\in E^{n,n-2}(\boldsymbol{C}^n)$. Thus

$$\int_\Gamma f\omega = \lim\int f\omega_j = \lim\int f\bar{\partial}\mu_j = 0.$$

References

1. F. Browder, *Functional analysis and partial differential equations*. II, Math. Ann. **145** (1961/62), 81–226. MR **25** #318.

2. J. Dieudonné and L. Schwartz, *La dualité dans les espaces* ($\mathfrak{F}$) *et* ($\mathfrak{L}\mathfrak{F}$), Ann. Inst. Fourier Grenoble **1** (1949), 61–101. MR **12**, 417.

3. L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N. J., 1966. MR **34** #2933.

4. J. L. Kelley and I. Namioka, *Linear topological spaces*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR **29** #3851.

5. L. Schwartz, *Théorie des distributions*. Tome I, Actualités Sci. Indust., no. 1091, Hermann, Paris, 1950. MR **12**, 31.

6. B. Weinstock, *Continuous boundary values of analytic functions of several complex variables*, Proc. Amer. Math. Soc. **21** (1969), 463–466. MR **38** #6106.

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