ON THE IDEAL STRUCTURE OF THE ALGEBRA OF RADIAL FUNCTIONS

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Abstract. Let $L$ denote the convolution Banach algebra of integrable functions defined on $\mathbb{R}^n$ and let $L_r$ consist of the sub-algebra of radial functions. If $I$ is a closed ideal of $L$, the zero-set of $I$ is defined by $Z(I) = \{ y | \hat{f}(y) = 0 \text{ for all } f \in I \}$ where $\hat{f}$ is the Fourier transform of $f$. The following theorem is proved. If $I_1$ and $I_2$ are closed ideals of $L_r$ such that $I_1 \subset I_2$ ($\subset$ denotes proper inclusion) then there is a closed ideal $I$ such that $I_1 \subset I \subset I_2$.

Let $n$ be a fixed positive integer, and let $L$ denote the Banach algebra of integrable functions defined on $\mathbb{R}^n$ with the usual norm and convolution. (The practice of identifying two functions which agree almost everywhere will be followed.) A function $f$ defined on $\mathbb{R}^n$ is said to be radial if $f(x) = \phi(|x|)$ for some function $\phi$ defined on $[0, \infty)$ and for almost every $x$ in $\mathbb{R}^n$; $L_r$ will denote the space of radial functions contained in $L$. A function in $L$ is radial if and only if its Fourier transform is a radial function (see [1, pp. 69-79]), so $L_r$ is a Banach algebra. If $I$ is a closed ideal of $L$ or of $L_r$, let $Z(I) = \{ y | \hat{f}(y) = 0 \text{ for every } f \in I \}$.

$Z(I)$ is called the zero-set of $I$.

Helson showed in [2] that if $I_1$ and $I_2$ are closed ideals of $L$ such that $Z(I_1) = Z(I_2)$ and $I_1 \subset I_2$ ($\subset$ denotes proper inclusion), then there is a closed ideal $I$ such that $I_1 \subset I \subset I_2$. In the present paper Helson's theorem will be used to prove the following:

Theorem. If $I_1$ and $I_2$ are closed ideals of $L_r$ such that $Z(I_1) = Z(I_2)$ and $I_1 \subset I_2$, then there is a closed ideal $I$ of $L_r$ such that $I_1 \subset I \subset I_2$.

The proof of the theorem will be given later; it is necessary, first, to examine how $L_r$ sits in $L$.

Let $d\mu$ be the positive measure of unit mass distributed uniformly on the hypersphere $S = \{ x | x \in \mathbb{R}^n \text{ and } |x| = 1 \}$, and set

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\[ f_r(x) = \int f(\|x\|y)\,d\mu(y). \]

The integral must exist for almost every \(x\) by Fubini's theorem since \(\mathbb{R}^n\) can be thought of as a product of two measure spaces: one being \(S\) with the measure \(d\mu\) and the other being \([0, \infty)\) with the measure \(cP^{n-1}dp\) where \(dp\) is Lebesgue measure and \(c\) is the surface area of \(S\). It also follows from Fubini's theorem that \(f_r\) is in \(L\). Define

\[ L_0 = \{ f \mid f \in L \text{ and } f_r(x) = 0 \text{ for almost every } x \text{ in } \mathbb{R}^n \}; \]

finally let \(f_0(x) = f(x) - f_r(x)\). The following lemmas list some properties of \(L_r, L_0, f_r\), and \(f_0\).

**Lemma 1.** The map \(f \to f_r\) is a continuous projection with unit norm, hence its null space \(L_0\) is closed and so \(L = L_0 \oplus L_r\).

The proof of Lemma 1 follows from the easily verified facts that \(\|f_r\| \leq \|f\|\) and that \(f = f_r\) if \(f\) is radial.

A thorough discussion of this decomposition can be found in [3].

**Lemma 2.** \(f\) is contained in \(L_0\) if and only if

\[ \int f(\rho y)\,d\mu(y) = 0 \quad (\rho > 0). \]

**Proof.** Application of Fubini's theorem yields

\[ \int f(\rho y)\,d\mu(y) = \int_{\mathbb{R}^n} f(x)\,dx \left\{ \int \exp(ix \cdot \rho y)\,d\mu(y) \right\} \]
\[ = \int_{\mathbb{R}^n} f(x)K(x)\,dx, \]

where \(K(x)\) is the value of the inner integral. \(K(x)\) is a radial function because \(\mu\) is a weak limit of radial functions and \(K(x)\) is the Fourier-Stieltjes transform of \(\mu\), or see [1, pp. 69–79]. Conversion of the last integral into hyperspherical coordinates yields (2).

To prove the converse, suppose (2) holds for some \(f\) in \(L\). Then

\[ \int f_r(\|x\|y)\,d\mu(y) + \int f_0(\|x\|y)\,d\mu(y) = 0. \]

The second integral vanishes by the first part of this lemma since \(f_0\) is in \(L_0\), and the value of the first integral is \(f_r(x)\). Thus \(f_r = 0\), so \(f_r = 0\) and hence \(f\) is in \(L_0\).
**Lemma 3.** The convolution of a function in $L_0$ and a function in $L_\infty$ is contained in $L_0$.

**Proof.** Suppose $f$ is in $L_\infty$ and $g$ is in $L_0$. Then for each $x$ in $\mathbb{R}^n$

$$
\int (f \star g)^\wedge(|x| y) d\mu(y) = \int f'(|x| y) \hat{g}(|x| y) d\mu(y)
$$

$$
= \hat{f}(x) \int \hat{g}(|x| y) d\mu(y)
$$

$$
= 0
$$

by Lemma 2.

**Lemma 4.** Let $I$ be a closed ideal of $L_\infty$ and let $K$ be the closed ideal of $L$ generated by $I$. Then $I = K \cap L_\infty$, and $Z(I) = Z(K)$.

**Proof.** $I$ is contained in $K$, hence in $K \cap L_\infty$. The fact that $K \cap L_\infty$ is contained in $I$ will follow from the stronger fact that if $f$ is in $K$, then $f_r$ is in $I$. Suppose

$$
(3) \quad f = h + \sum_{i=1}^{m} h_i \ast g_i \quad (h \in I, \ h_i \in I, \ g_i \in L; \ i = 1, 2, \ldots, m).
$$

Then $f = h + \sum_{i=1}^{m} h_i \ast (g_i)_r + \sum_{i=1}^{m} h_i \ast (g_i)_0$. The second sum is contained in $L_0$ by Lemma 3 and the first sum is contained in $I$ because $I$ is an ideal of $L_\infty$. Finally, the first sum plus $h$ is $f_r$ by Lemma 1; hence if $f$ has the form (3) then $f_r$ is in $I$. If $f$ is any function in $K$, there is a sequence $\{f_k\}$ of finite linear combinations of the form of (3) such that $f_k$ converges to $f$ in $L$. The transformation of $f$ into $f_r$ is continuous on $L$ so $\{(f_k)_r\}$ converges to $f_r$. Since $I$ is closed, it must contain $f_r$. Finally $Z(K) \subseteq Z(I)$ because $I \subseteq K$ and $Z(K) = Z(I)$ since finite linear combinations of the form of (3) are dense in $K$.

**Proof of Theorem.** Let $K_1$ and $K_2$ be the closed ideals of $L$ generated by $I_1$ and $I_2$ respectively. Then $K_1 \subseteq K_2$ by Lemma 4 because $K_1 \cap L_\infty = I_1 \subseteq I_2 = K_2 \cap L_\infty$, and

$$
Z(K_1) = Z(I_1) = Z(I_2) = Z(K_2).
$$

By Helson's theorem there must be a closed ideal $K$ such that $K_1 \subseteq K \subseteq K_2$. Since $K_2$ is the ideal generated by $I_2$, it follows that $K \cap L_\infty \subseteq I_2$. The inclusion $I_1 \subseteq K \cap L_\infty$ is not immediate. Suppose there is no closed ideal $K$ of $L$ such that

$$
(4) \quad I_1 \subseteq K \cap L_\infty \subseteq I_2,
$$
then define $\mathcal{K}$ to be the collection of all closed ideals of $L$ such that

$$K_1 \subseteq K \subseteq K_2 \quad \text{and} \quad K \cap L_r = I_1.$$ 

Let $\mathcal{K}$ be ordered by inclusion and let $K^*$ be the union of all the ideals in a maximal chain of $\mathcal{K}$.

$K^*$ is contained in $\mathcal{K}$. To see this let $J$ be the closure in $L$ of $K^*$. If $f$ is in $K^* \cap L_r$, then $f$ is in $K \cap L_r$ for some $K$ in $\mathcal{K}$ so $f$ is in $I_1$; thus $K^* \cap L_r = I_1$ so $J \cap L_r = I_1$. Since $J \cap L_r = I_1$, it follows that $J \subseteq K_2$. Since $K^*$ is a union of elements of $\mathcal{K}$ it follows that $K_1 \subseteq J \subseteq K_2$. Thus $J$ is in $\mathcal{K}$ and so $K^* = J$ by the construction of $K^*$; hence, $K^*$ is in $\mathcal{K}$. It also follows that $Z(K^*) = Z(K_2)$ because $K^*$ lies between $K_1$ and $K_2$.

Helson's theorem can now be invoked to guarantee the existence of an ideal $K^{**}$ such that $K^* \subseteq K^{**} \subseteq K_2$. Since $K^{**} \subseteq K_2$ it follows that $K^{**} \cap L_r \subseteq I_2$, and the proper inclusion $I_1 \subseteq K^{**} \cap L_r$ holds by the construction of $K^*$. Thus $K^{**}$ contradicts our assumption that no ideal of $L$ satisfies (4).

REFERENCES