A COMBINATORIAL PROBLEM AND CONGRUENCES FOR THE RAYLEIGH FUNCTION

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ABSTRACT. Let z be a positive integer and let m be the number of nonzero terms in the base 2 expansion of z. Define $f(z, s)$ as the number of positive integers $r \leqq z/2$ such that the number of nonzero terms in the base 2 expansion of r plus the number of nonzero terms in the base 2 expansion of $z-r$ is equal to $m+s$. We find formulas for $f(z, s)$ and show how these formulas can be used in proving congruences for the Rayleigh function.

1. **Introduction.** We shall assume that

$$z = 2^{e_1} + \cdots + 2^{e_m}, \qquad e_1 > \cdots > e_m \geqq 0.$$

For $s \geqq 0$ define $f(z, s)$ as the number of positive integers $r \leqq z/2$ such that if

$$\begin{aligned} r &= 2^{b_1} + \cdots + 2^{b_v}, & b_1 &> \cdots > b_v \geqq 0, \\ z - r &= 2^{c_1} + \cdots + 2^{c_w}, & c_1 &> \cdots > c_w \geqq 0, \end{aligned} \tag{1.1}$$

then $v+w=m+s$.

In §4 we point out that $f(z, s)$ is equal to the number of binomial coefficients $\binom{z}{r}$, $0<r\leqq z/2$, divisible by 2^s but not by 2^{s+1}. The main purpose of this paper is to consider the problem of evaluating $f(z, s)$ for $s \geqq 0$ and to show how this function can be used for proving congruences for the Rayleigh function, $\sigma_{2n}(\nu)$. We note that in [2] it was proved that $f(z, 0) = 2^{m-1}-1$.

The Rayleigh function has been the subject of a number of papers by Kishore (see [3] and [4] for example). It can be defined by means of the recurrence formula

$$(\nu + n)\sigma_{2n}(\nu) = \sum_{i=1}^{n-1} \sigma_{2i}(\nu)\sigma_{2n-2i}(\nu), \tag{1.2}$$

where $\sigma_2(\nu) = 1/4(\nu+1)$. It is known that

$$\begin{aligned} \sigma_{2n}(1/2) &= (-1)^{n-1}2^{2n-1}B_{2n}/(2n)!, \\ \sigma_{2n}(-1/2) &= (-1)^n 2^{2n-2}G_{2n}/(2n)! \end{aligned}$$

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where B_{2n} is the $2n$th Bernoulli number and $G_n = 2(1-2^n)B_n$.

In this paper we prove congruences (mod 8) and (mod 16) for $2^y\sigma_{2n}(a/b)$, when neither the numerator nor denominator of $2^y\sigma_{2n}(a/b)$ is divisible by 2.

2. **A combinatorial problem.** The notation of the introduction will be used in this section, and we will make use of the function $\alpha(j)$ defined by means of

$$\begin{aligned}\alpha(j) &= 2 \quad \text{if } s+m = 2j,\\ &= 1 \quad \text{if } s+m > 2j.\end{aligned}$$

THEOREM 2.1. *If $e_i - e_{i+1} > s > 0$ for $i = 1, \cdots, m-1$ and $e_m \geqq s$, then*

$$f(z, s) = \sum_{j=1}^{m} \binom{m}{j}\binom{s-1}{j-1}\alpha(j)2^{s+m-2j-1}.$$

PROOF. To find r and $z-r$ satisfying (1.1) we first pick j elements from $e_1, \cdots, e_m$. Call them $d_1, \cdots, d_j$. Consider any composition of s into j parts (see [5, p. 107]):

$$s = s_1 + \cdots + s_j.$$

Then for $i = 1, \cdots, j$ we have

$$2^{d_i} = 2^{d_i - s_i} + \sum_{h=0}^{s_i-1} 2^{d_i - s_i + h}.$$

The exponents b and c of (1.1) will be chosen from the $s+j$ numbers $d_i - s_i + h$ and the $m-j$ e's which were not picked. There are 2^{m+s-2j} ways to divide these numbers between r and $z-r$. There are $\binom{m}{j}$ ways to pick $d_1, \cdots d_j$ and there are $\binom{s-1}{j-1}$ compositions of s into j parts [5, p. 124]. We divide our final answer by 2 (except when $s+m=2j$) to eliminate $r > z/2$.

THEOREM 2.2. *If $e_i - e_{i+1} > s > 0$ for q terms e_i, if $e_i - e_{i+1} = s$ for $m-q-1$ terms and $e_m \geqq s$, then*

$$f(z, s) = (q+1)2^{s+m-3} + \sum_{j=2}^{m} \binom{m}{j}\binom{s-1}{j-1}\alpha(j)2^{s+m-2j-1}.$$

THEOREM 2.3. *If $e_i - e_{i+1} > s > 0$ for q terms e_i, if $e_i - e_{i+1} = s$ for $m-q-1$ terms and $e_m = u$, $0 \leqq u < s$, then*

$$f(z, s) = q2^{s+m-3} + \sum_{j=2}^{m} \binom{m}{j}\binom{s-1}{j-1}\alpha(j)2^{s+m-2j-1}$$
$$- \sum_{j=2}^{s-u} \binom{m-1}{j-1}\binom{s-u-1}{j-1}\alpha(j)2^{s+m-2j-1}.$$

PROOF. We use the method of Theorem 2.1. Pick j terms from $e_1, \cdots, e_m$ and call them $d_1, \cdots, d_j$. The term $q\,2^{s+m-3}$ in the formula corresponds to the case $j=1$. For $2 \leqq j \leqq m$ we proceed exactly as we did in Theorem 2.1. We observe, however, that certain compositions of s into j parts will not be allowed when $2 \leqq j \leqq s-u$ and e_m is picked as one of the d's. When this happens, there are $\binom{m-1}{j-1}$ ways of picking the remaining d's, and we need to find the number of compositions of s into j parts $s_1 + \cdots + s_j$ such that $s_1 > u$. An enumerating generating function for such compositions is

$$(x^{u+1} + x^{u+2} + \cdots)(x + x^2 + x^3 + \cdots)^{j-1}.$$

From this we determine the number of such compositions to be $\binom{s-u-1}{j-1}$. The formula of Theorem 2.3 now follows.

3. **Congruences for the Rayleigh function.** We now show how the formulas of §2 can be used in proving congruences for the Rayleigh function.

Let a/b be a rational number, a odd, b even, $b=(2k+1)2^t$, and let $y=2n+1+(1-2n)t-m$, where m is the number of nonzero terms in the base 2 expansion of $2n$. In [2] the following results were proved.

LEMMA 3.1. *If* $2n=2^{e_1}+\cdots+2^{e_m}$, $e_1>\cdots>e_m \geqq 1$, *then*

$$2^y\sigma_{2n}(a/b) \equiv 1 \pmod 2.$$

LEMMA 3.2. *If* $2n=2^{e_1}$, $e_1>2$, *then*

$$2^y\sigma_{2n}(a/b) \equiv 5(2k+1)/a \pmod 8.$$

LEMMA 3.3. *If* $2n=2^{e_1}+\cdots+2^{e_m}$, $e_i-e_{i+1}>1$ *for* $i=1, \cdots, m-1$ *and* $e_m \geqq 1$, *then*

$$2^y\sigma_{2n}(a/b) \equiv (-1)^{m+1}(2k+1)/a \pmod 4.$$

We can now prove the following theorems.

THEOREM 3.1. *If* $2n=2^{e_1}+2^{e_2}$, $e_1-e_2>2$, $e_2>2$, *then*

$$2^y\sigma_{2n}(a/b) \equiv 7(2k+1)/a \pmod 8.$$

PROOF. From (1.2) and Lemmas 3.1, 3.2 and 3.3, we have

$$2^{\nu}\sigma_{2n}(a/b) \equiv [f(2n, 0) + 6f(2n, 1) + 4(f(2n, 2) - 1) + 2](2k + 1)/a$$
$$\equiv 7(2k + 1)/a \pmod 8.$$

THEOREM 3.2. *If* $2n = 2^{e_1} + 2^{e_2} + 2^{e_3}$, $e_i - e_{i+1} > 2$ *for* $i = 1, 2$ *and* $e_3 > 2$, *then*

$$2^{\nu}\sigma_{2n}(a/b) \equiv 5(2k + 1)/a \pmod 8.$$

PROOF. We have

$$2^{\nu}\sigma_{2n}(a/b) \equiv [3f(2n, 0) + 2f(2n, 1) + 4f(2n, 2) + 4](2k + 1)/a$$
$$\equiv 5(2k + 1)/a \pmod 8.$$

THEOREM 3.3. *If* $2n = 2^{e_1} + \cdots + 2^{e_m}$, $e_i - e_{i+1} > 2$ *for* $i = 1, \cdots, m-1$ *and* $e_m > 2$, *then*

$$2^{\nu}\sigma_{2n}(a/b) \equiv 7(2k + 1)/a \pmod 8 \quad \textit{if } m \textit{ is even},$$
$$\equiv 5(2k + 1)/a \pmod 8 \quad \textit{if } m \textit{ is odd}.$$

PROOF. The proof is by induction on m. The theorem is true for $m = 1, 2, 3$. Assume it is true for $1, 2, 3, \cdots, m-1$. If m is even we have

$$2^{\nu}\sigma_{2n}(a/b) \equiv [f(2n, 0) + 6f(2n, 1) + 4f(2n, 2)](2k + 1)/a \pmod 8.$$

We know

$$f(2n, 0) = 2^{m-1} - 1 \equiv 7 \pmod 8,$$
$$f(2n, 1) = m2^{m-2} \equiv 0 \pmod 4,$$
$$f(2n, 2) = m2^{m-1} + \binom{m}{2} 2^{m-3} \equiv 0 \pmod 2.$$

Therefore

$$2^{\nu}\sigma_{2n}(a/b) \equiv 7(2k + 1)/a \pmod 8.$$

If m is odd we have

$$2^{\nu}\sigma_{2n}(a/b) \equiv [3f(2n, 0) + 2f(2n, 1) + 4f(2n, 2)](2k + 1)/a$$
$$\equiv 5(2k + 1)/a \pmod 8.$$

We can use the method of this section to prove congruences (mod 16). The following theorem is first verified for $m = 1, 2, 3, 4$ and then proved by induction on m.

THEOREM 3.4. *If* $2n = 2^{e_1} + \cdots + 2^{e_m}$, $e_i - e_{i+1} > 3$ *for* $i = 1, \cdots, m-1$ *and* $e_m > 3$, *then*

$$\begin{aligned} 2^{\nu}\sigma_{2n}(a/b) &\equiv 5((2k+1)/a)^3 \pmod{16} \quad \text{if } m \equiv 3 \pmod 4, \\ &\equiv 7((2k+1)/a)^3 \pmod{16} \quad \text{if } m \equiv 2 \pmod 4, \\ &\equiv 13((2k+1)/a)^3 \pmod{16} \quad \text{if } m \equiv 1 \pmod 4, \\ &\equiv 15((2k+1)/a)^3 \pmod{16} \quad \text{if } m \equiv 0 \pmod 4. \end{aligned}$$

4. **Relationship between $f(z, s)$ and binomial coefficients.** We use the notation of the introduction in this section. Consider the binomial coefficient $\binom{z}{r}$, $0<r\leqq z/2$. It is well known that the exponent of the highest power of 2 dividing $\binom{z}{r}$ is $s=v+w-m$. This follows from the fact that if

$$n = 2^{a_1} + \cdots + 2^{a_i}, \quad a_1 > \cdots > a_i \geqq 0,$$

then the exponent of the highest power of 2 dividing $n!$ is $n-i$ [1, p. 61]. Hence, by the definition of $f(z, s)$, we see that $f(z, s)$ is equal to the number of binomial coefficients $\binom{z}{r}$, $0<r\leqq z/2$, divisible by 2^s but not by 2^{s+1}. I am grateful to the referee and to L. Carlitz for suggestions concerning this section.

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