THE COHOMOLOGY RING OF A FINITE GROUP SCHEME

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ABSTRACT. Let k be a field and let A be a k-algebra with additional structure so that Spec A is a finite commutative group scheme over k, (so A is a Hopf algebra). Let $H^{\cdot}(A, k)$ be the Hochschild cohomology ring. In another paper, we demonstrated that if k is a perfect field:

(a) $H^{\cdot}(A, k)$ is generated by H^1 and H^2_{sym}.

(b) If characteristic $k = p \neq 2$, then $H^{\cdot}(A, k)$ is freely generated by H^1 and H^2_{sym}.

(c) If characteristic $k = 2$, then there are subspaces V_1, V_2 of H^1 and V_3 of H^2_{sym} such that $H^{\cdot}(A, k)$ is generated by V_1, V_2, V_3 and the only relations are $f^2 = 0$ for all f in V_1.

In this paper we show that if k is arbitrary (a) and (b) still hold, and we use an example of Oort and Mumford to show that (c) does not hold for arbitrary k.

1. **$H^{\cdot}(A, k)$ for k a field of characteristic $p \neq 2$.** Recall briefly the definition of a finite group scheme. Let A be a k-algebra, A finitely generated as a k-module (k is always a field in this paper). We have maps $\Delta: A \to A \otimes A$, $s: A \to A$, $\epsilon: A \to k$ satisfying the usual axioms, [2] e.g. $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta: A \to A \otimes A \otimes A$.

We also assume Δ is commutative. This means that if $T: A \otimes A \to A \otimes A$ is defined by $T(a \otimes b) = b \otimes a$, then $T \circ \Delta = \Delta$. Then Spec A together with Δ, s, ϵ is a finite group scheme. (In other words A is a Hopf algebra over k.)

Let K be an extension of k. Let $A_K = A \otimes_k K$. Then Spec A_K together with maps $\Delta \otimes 1$, $s \otimes 1$, $\epsilon \otimes 1$ is a finite group scheme over K.

Let $H^{\cdot}(A, k)$ be the Hochschild cohomology ring [1, p. 169], with product the cup product induced by $\Delta: A \to A \otimes A$. Then $H^{\cdot}(A, k)$ is a commutative graded ring; meaning if f is in H^n, g in H^m, then $f \cup g = (-1)^{mn} g \cup f$.

Recall that if $C^{\cdot}(A, k)$ is the standard resolution of k, [1, p. 174], then $C^2(A, k) \cong \mathrm{Hom}_k(A^{\otimes 2}, k)$. We define $H^2_{\mathrm{sym}}(A, k)$ as the subgroup of $H^2(A, k)$ generated by the standard cocycles f such that $f(x, y) = f(y, x)$ for all x, y in A.

In [2] the following theorem is proved.

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THEOREM 1 [2, p. 317, THEOREM 2.2]. *If* Spec A *is a commutative group scheme over a perfect field* k, *then*

(a) $H^{\cdot}(A, k)$ *is generated by* H^1 *and* H^2_{sym}.

(b) *If characteristic* $(k) \neq 2$, *then* $H^{\cdot}(A, k)$ *is freely generated by* H^1 *and* H^2_{sym}. *Freely generated means the only relations are* $f \cup g = (-1)^{mn} g \cup f$ *for* f *in* H^n, g *in* H^m.

(c) *If characteristic* $(k) = 2$, *then there exist subspaces* V_1, V_2, *of* H^1, *and* V_3 *of* H^2_{sym} *so that* $H^{\cdot}(A, k)$ *is generated by* V_1, V_2, V_3 *with additional relations* $f \cup f = 0$ *for all* f *in* V_1.

PROPOSITION 1. *Let* Spec A *be a finite group scheme over a field* k; *let* K *be an extension field of* k. *Let* $A_K = A \otimes_k K$ *as above.*

Then there is a natural isomorphism

$$H^{\cdot}(A, k) \otimes_k K \to H^{\cdot}(A_K, K).$$

PROOF. This is just another form of the universal coefficient theorem. It is proven by noting that if $C.(A, k)$ is an A-free resolution of the A-module k, then $C.(A, k) \otimes K = C.(A_K, K)$ is an A_K free resolution of the A_K module K. Also we have

$$\begin{aligned} \mathrm{Hom}_A(C_n(A, k), k) \otimes K &\cong \mathrm{Hom}_{A \otimes_k K}(C_n(A, k) \otimes K, K) \\ &= \mathrm{Hom}_{A_K}(C_n(A_K, K), K). \end{aligned}$$

Taking cohomology we get the required isomorphism of cohomology groups. One must also check that this is a homomorphism for the ring structure but this is just a matter of checking the maps induced by Δ and $\Delta \otimes 1$.

Note that we have an injection $\phi: H^{\cdot}(A, k) \to H^{\cdot}(A, k) \otimes_k K \cong H^{\cdot}(A_K, K)$.

THEOREM 2. *Let* Spec A *be a finite group scheme over a field* k. *Then* $H^{\cdot}(A, k)$ *is generated by* H^1 *and* H^2_{sym}. *Moreover* $H^{\cdot}(A, k)$ *is freely generated by* H^1 *and* H^2_{sym} *if characteristic* $(k) \neq 2$.

PROOF. Let K be a perfect extension of k. We have $\phi: H^{\cdot}(A, k) \to H^{\cdot}(A, k) \otimes K$. We identify $H^{\cdot}(A, k) \otimes K$ with $H^{\cdot}(A_K, K)$. Note that $H^2_{\mathrm{sym}}(A, k) \otimes K$ will then be identified with $H^2_{\mathrm{sym}}(A_K, K)$.

Let V_n be the subspace of $H^n(A, k)$ generated by H^1 and H^2_{sym}. By Theorem 1, $V_n \otimes_k K = H^n(A, k) \otimes_k K$ and so $V_n = H^n(A, k)$.

Now let characteristic $(k) \neq 2$. By part (b) of Theorem 1, there are no relations on $H^1(A_K, K)$ and $H^2_{\mathrm{sym}}(A_K, K)$. Since

$$\phi: H^{\cdot}(A, k) \to H^{\cdot}(A_K, K)$$

is an injection, there cannot be any relations on $H^1(A, k)$ and $H^2_{\mathrm{sym}}(A, k)$.

2. **An example for characteristic** 2. In [**3**, p. 320], Oort and Mumford give an example of a finite group scheme which is not of "truncation type." We are interested in this example only when $p=2$, and then it becomes:

EXAMPLE. Let k be a field of characteristic 2. Let $a\in k$, $a^{1/2}\notin k$. Let

$$A = k[x, y]/(x^4, x^2 - ay^2)$$

with $\Delta: A\to A\otimes A$ defined by $\Delta x=1\otimes x+x\otimes 1$, $\Delta y=1\otimes y+y\otimes 1$. s and ϵ are the obvious maps.

Then Spec A is a finite group scheme over k.

PROPOSITION 2. *If A is as above, $H^\cdot(A, k)$ is a commutative graded ring isomorphic to*

$$k[V_1, V_2, U]/(aV_1^2 - V_2^2),$$

V_1, V_2 of degree 1, *U of degree* 2.

PROOF. Let $K=k(a^{1/2})=k(b)$ with $b^2=a$. Let $A_K=A\otimes_k K$ as before and let $x'=x\otimes 1$, $y'=y\otimes 1$. Then

$$A_K = K[x', y']/(x'^4, x'^2 - ay'^2) \cong K[x', z]/(x'^4, z^2)$$

where $z=x'-by'$.

We will want to define resolutions $C.(A, k)$ of the A module k and $C.(A_K, K)$ of the A_K module K. We will use primes to denote the elements of $C.(A_K, K)$. Let T_1, T_2, T_1', T_2' be of degree 1.

We want $\partial T_1=x$, $\partial T_2=y$, $\partial T_1'=x'$, $\partial T_2'=z$ so we can choose $T_1'=T_1\otimes 1$, $T_2'=T_1\otimes 1+T_2\otimes b$.

Let S_1, S_2, S_1', S_2' be of degree 2. We want

$$\partial S_1 = x^3T_1, \quad \partial S_2 = xT_1 - ayT_2, \quad \partial S_1' = x'^3T_1', \quad \partial S_2' = zT_2'.$$

Now let $C.(A, k)$ be the algebra which is the exterior algebra on T_1, T_2 tensored with the divided power algebra on S_1, S_2. Let $C.(A_K, K)$ be the same thing with T_1', T_2', S_1', S_2' replacing T_1, T_2, S_1, S_2 respectively. We know $C.(A_K, K)$ is a free A_K resolution of K, [**2**, p. 315]. Also we can let $S_1'=S_1\otimes 1$ and $S_2'=S_2\otimes 1+T_1T_2\otimes b$. Then it is clear that $C.(A, k)\otimes_k K$ can be identified with $C.(A_K, K)$. This means that $C.(A, k)$ is a free resolution also.

Take $C^\cdot(A, k) = \mathrm{Hom}_A(C.(A, k), k)$ and $C^\cdot(A_K, K) = \mathrm{Hom}_{A_K}(C.(A_K, K), K)$. So $C^1(A, k) \cong \mathrm{Hom}_A(A^{\otimes 2}, k) \cong \mathrm{Hom}_k(k^{\otimes 2}, k)$.

Let V_1, V_2 be the dual basis to T_1, T_2 under the above identification so $V_1(T_1)=1$, etc. Then V_1V_2 will be dual to T_1T_2 in $C^2(A, k)$. Complete the dual basis by letting U_1, U_2 be k dual to S_1, S_2. Do the same for the primes. Then

$$C^{\cdot}(A_K, K) \cong K[V_1', V_2', U_1', U_2']/(V_1'^2, V_2'^2 - U_2')$$

with $\delta V_i' = 0 = \delta U_i'$. See [2, p. 317, Corollary].

We also have a natural map identifying $C^{\cdot}(A_K, K)$ with $C^{\cdot}(A, k) \otimes K$. We have

$$\begin{pmatrix} T_1' \\ T_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & b \end{pmatrix} \begin{pmatrix} T_1 \otimes 1 \\ T_2 \otimes 1 \end{pmatrix}.$$

Dualizing we get

$$\begin{pmatrix} V_1 \otimes 1 \\ V_2 \otimes 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}.$$

Also

$$\begin{pmatrix} S_1' \\ S_2' \\ T_1' T_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} S_1 \otimes 1 \\ S_2 \otimes 1 \\ T_1 T_2 \otimes 1 \end{pmatrix}$$

induces

$$\begin{pmatrix} U_1 \otimes 1 \\ U_2 \otimes 1 \\ V_1 V_2 \otimes 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & b \end{pmatrix} \begin{pmatrix} U_1' \\ U_2' \\ V_1' V_2' \end{pmatrix}.$$

On the cohomology level, we have $V_1'^2 = 0$, $V_2'^2 = U_2'$. However $V_1 \otimes 1 = V_1' + V_2'$ which implies $V_1^2 \otimes 1 = U_2'$. Similarly $V_2 \otimes 1 = b V_2'$ implies $V_2^2 \otimes 1 = b^2 U_2' = a U_2'$. Thus $a V_1^2 = V_2^2$. Moreover $V_1^2 = U_2$. Let $U = U_1$. Our result follows since $H^{\cdot}(A, k) \otimes_k K = H^{\cdot}(A_K, K)$ and so no more relations are possible.

References

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