THE COHOMOLOGY RING OF A FINITE GROUP SCHEME

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Abstract. Let $k$ be a field and let $A$ be a $k$-algebra with additional structure so that Spec $A$ is a finite commutative group scheme over $k$, (so $A$ is a Hopf algebra). Let $H^*(A, k)$ be the Hochschild cohomology ring. In another paper, we demonstrated that if $k$ is a perfect field:

(a) $H^*(A, k)$ is generated by $H^1$ and $H^2_{\text{sym}}$.

(b) If characteristic $k = p \neq 2$, then $H^*(A, k)$ is freely generated by $H^1$ and $H^2_{\text{sym}}$.

(c) If characteristic $k = 2$, then there are subspaces $V_1, V_2$ of $H^1$ and $V_3$ of $H^2_{\text{sym}}$ such that $H^*(A, k)$ is generated by $V_1, V_2, V_3$ and the only relations are $f^2 = 0$ for all $f$ in $V_1$.

In this paper we show that if $k$ is arbitrary (a) and (b) still hold, and we use an example of Oort and Mumford to show that (c) does not hold for arbitrary $k$.

1. $H^*(A, k)$ for $k$ a field of characteristic $p \neq 2$. Recall briefly the definition of a finite group scheme. Let $A$ be a $k$-algebra, $A$ finitely generated as a $k$-module ($k$ is always a field in this paper). We have maps $\Delta : A \to A \otimes A$, $s : A \to A$, $\epsilon : A \to k$ satisfying the usual axioms, [2] e.g. $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : A \otimes A \to A \otimes A \otimes A$.

We also assume $\Delta$ is commutative. This means that if $T : A \otimes A \to A \otimes A$ is defined by $T(a \otimes b) = b \otimes a$, then $T \circ \Delta = \Delta$. Then Spec $A$ together with $\Delta, s, \epsilon$ is a finite group scheme. (In other words $A$ is a Hopf algebra over $k$.)

Let $K$ be an extension of $k$. Let $A_K = A \otimes_k K$. Then Spec $A_K$ together with maps $\Delta \otimes 1, s \otimes 1, \epsilon \otimes 1$ is a finite group scheme over $K$.

Let $H^*(A, k)$ be the Hochschild cohomology ring [1, p. 169], with product the cup product induced by $\Delta : A \to A \otimes A$. Then $H^*(A, k)$ is a commutative graded ring; meaning if $f$ is in $H^n$, $g$ in $H^m$, then $f \cup g = (-1)^{mn} g \cup f$.

Recall that if $C^*(A, k)$ is the standard resolution of $k$, [1, p. 174], then $C^*(A, k) \cong \text{Hom}_k(A^{\otimes 2}, k)$. We define $H^2_{\text{sym}}(A, k)$ as the subgroup of $H^2(A, k)$ generated by the standard cocycles $f$ such that $f(x, y) = f(y, x)$ for all $x, y$ in $A$.

In [2] the following theorem is proved.

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Theorem 1 [2, p. 317, Theorem 2.2]. If Spec $A$ is a commutative group scheme over a perfect field $k$, then

(a) $H'(A, k)$ is generated by $H^1$ and $H^2_{\text{sym}}$.

(b) If characteristic $(k) \neq 2$, then $H'(A, k)$ is freely generated by $H^1$ and $H^2_{\text{sym}}$. Freely generated means the only relations are $f \cup g = (-1)^{m+n}f \cup g$ for $f, g$ in $H^n$.

(c) If characteristic $(k) = 2$, then there exist subspaces $V_1, V_2$ of $H^1$, and $V_3$ of $H^2_{\text{sym}}$ so that $H'(A, k)$ is generated by $V_1, V_2, V_3$ with additional relations $f \cup f = 0$ for all $f$ in $V_1$.

Proposition 1. Let Spec $A$ be a finite group scheme over a field $k$; let $K$ be an extension field of $k$. Let $A_K = A \otimes_k K$ as above.

Then there is a natural isomorphism

$$H'(A, k) \otimes_k K \rightarrow H'(A_K, K).$$

Proof. This is just another form of the universal coefficient theorem. It is proven by noting that if $C_\ast(A, k)$ is an $A$-free resolution of the $A$-module $k$, then $C_\ast(A, k) \otimes K = C_\ast(A_K, K)$ is an $A_K$ free resolution of the $A_K$ module $K$. Also we have

$$\text{Hom}_A(C_n(A, k) \otimes K) \cong \text{Hom}_{A \otimes_k K}(C_n(A, k) \otimes K, K) = \text{Hom}_{A_K}(C_n(A_K, K), K).$$

Taking cohomology we get the required isomorphism of cohomology groups. One must also check that this is a homomorphism for the ring structure but this is just a matter of checking the maps induced by $\Delta$ and $\Delta \otimes 1$.

Note that we have an injection $\phi: H'(A, k) \rightarrow H'(A, k) \otimes_k K \cong H'(A_K, K)$.

Theorem 2. Let Spec $A$ be a finite group scheme over a field $k$. Then $H'(A, k)$ is generated by $H^1$ and $H^2_{\text{sym}}$. Moreover $H'(A, k)$ is freely generated by $H^1$ and $H^2_{\text{sym}}$ if characteristic $(k) \neq 2$.

Proof. Let $K$ be a perfect extension of $k$. We have $\phi: H'(A, k) \rightarrow H'(A, k) \otimes K$. We identify $H'(A, k) \otimes K$ with $H'(A_K, K)$. Note that $H^2_{\text{sym}}(A, k) \otimes K$ will then be identified with $H^2_{\text{sym}}(A_K, K)$.

Let $V_n$ be the subspace of $H^n(A, k)$ generated by $H^1$ and $H^2_{\text{sym}}$. By Theorem 1, $V_n \otimes_k K = H^n(A, k) \otimes_k K$ and so $V_n = H^n(A, k)$.

Now let characteristic $(k) \neq 2$. By part (b) of Theorem 1, there are no relations on $H^1(A_K, K)$ and $H^2_{\text{sym}}(A_K, K)$. Since

$$\phi: H'(A, k) \rightarrow H'(A_K, K)$$

is an injection, there cannot be any relations on $H^1(A, k)$ and $H^2_{\text{sym}}(A, k)$.
2. An example for characteristic 2. In [3, p. 320], Oort and Mumford give an example of a finite group scheme which is not of “truncation type.” We are interested in this example only when \( p = 2 \), and then it becomes:

Example. Let \( k \) be a field of characteristic 2. Let \( a \in k \), \( a^{1/2} \in k \).

Let

\[
A = k[x, y]/(x^4, x^2 - ay^2)
\]

with \( \Delta : A \to A \otimes A \) defined by \( \Delta x = 1 \otimes x + x \otimes 1 \), \( \Delta y = 1 \otimes y + y \otimes 1 \). \( s \) and \( \epsilon \) are the obvious maps.

Then \( \text{Spec } A \) is a finite group scheme over \( k \).

Proposition 2. If \( A \) is as above, \( H^r(A, k) \) is a commutative graded ring isomorphic to

\[
k[V_1, V_2, U]/(aV_1^2 - V_2^2),
\]

\( V_1, V_2 \) of degree 1, \( U \) of degree 2.

Proof. Let \( K = k(a^{1/2}) = k(b) \) with \( b^2 = a \). Let \( A_K = A \otimes_k K \) as before and let \( x' = x \otimes 1 \), \( y' = y \otimes 1 \). Then

\[
A_K = K[x', y']/(x'^4, x'^2 - ay'^2) \cong K[x', z]/(x'^4, z^2)
\]

where \( z = x' - by' \).

We will want to define resolutions \( C_*(A, k) \) of the \( A \) module \( k \) and \( C_*(A_K, K) \) of the \( A_K \) module \( K \). We will use primes to denote the elements of \( C_*(A_K, K) \). Let \( T_1, T_2, T_1', T_2' \) be of degree 1.

We want \( \partial T_1 = x \), \( \partial T_2 = y \), \( \partial T_1' = x' \), \( \partial T_2' = z \) so we can choose \( T_1' = T_1 \otimes 1 \), \( T_2' = T_1 \otimes 1 + T_2 \otimes b \).

Let \( S_1, S_2, S_1', S_2' \) be of degree 2. We want

\[
\partial S_1 = x^2 T_1, \quad \partial S_2 = x T_1 - ay T_2, \quad \partial S_1' = x'^2 T_1', \quad \partial S_2' = z T_2'.
\]

Now let \( C_*(A, k) \) be the algebra which is the exterior algebra on \( T_1, T_2 \) tensored with the divided power algebra on \( S_1, S_2 \). Let \( C_*(A_K, K) \) be the same thing with \( T_1', T_2', S_1', S_2' \) replacing \( T_1, T_2, S_1, S_2 \) respectively. We know \( C_*(A_K, K) \) is a free \( A_K \) resolution of \( K \), [2, p. 315]. Also we can let \( S_1' = S_1 \otimes 1 \) and \( S_2' = S_2 \otimes 1 + T_1 T_2 \otimes b \). Then it is clear that \( C_*(A, k) \otimes_k K \) can be identified with \( C_*(A_K, K) \). This means that \( C_*(A, k) \) is a free resolution also.

Take \( C^*(A, k) = \text{Hom}_A(C_*(A, k), k) \) and \( C^*(A_K, K) = \text{Hom}_{A_K}(C_*(A_K, K), K) \). So \( C^*(A, k) \cong \text{Hom}_A(A \otimes_k k, k) \cong \text{Hom}_k(k \otimes_k k) \).

Let \( V_1, V_2 \) be the dual basis to \( T_1, T_2 \) under the above identification so \( V_1(T_1) = 1 \), etc. Then \( V_1 V_2 \) will be dual to \( T_1 T_2 \) in \( C^2(A, k) \). Com-
plete the dual basis by letting \( U_1, U_2 \) be \( k \) dual to \( S_1, S_2 \). Do the same for the primes. Then

\[
C'(A_k, K) \cong K[V'_1, V'_2, U'_1, U'_2]/(V'^2_1, V'^2_2 - U'_2)
\]

with \( \delta V'_1 = 5 U'_1 \). See [2, p. 317, Corollary].

We also have a natural map identifying \( C(A_k, K) \) with \( C(A, k) \otimes K \). We have

\[
\begin{pmatrix} T'_1 \\ T'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & b \end{pmatrix} \begin{pmatrix} T_1 \otimes 1 \\ T_2 \otimes 1 \end{pmatrix}.
\]

Dualizing we get

\[
\begin{pmatrix} V_1 \otimes 1 \\ V_2 \otimes 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}.
\]

Also

\[
\begin{pmatrix} S'_1 \\ S'_2 \\ T'_1 T'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} S_1 \otimes 1 \\ S_2 \otimes 1 \\ T_1 T_2 \otimes 1 \end{pmatrix}
\]

induces

\[
\begin{pmatrix} U_1 \otimes 1 \\ U_2 \otimes 1 \\ V_1 V_2 \otimes 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & b \end{pmatrix} \begin{pmatrix} U'_1 \\ U'_2 \\ V'_1 V'_2 \end{pmatrix}.
\]

On the cohomology level, we have \( V'^2_1 = 0, V'^2_2 = U'_2 \). However \( V_1 \otimes 1 = V'_1 + V'_2 \) which implies \( V'^2_1 \otimes 1 = U'_2 \). Similarly \( V_2 \otimes 1 = b V'_2 \) implies \( V'^2_2 \otimes 1 = b^2 U'_2 = a U'_2 \). Thus \( a V'^2_1 = V'^2_2 \). Moreover \( V'^2_2 = U_2 \). Let \( U = U_1 \). Our result follows since \( H^*(A, k) \otimes_k K = H^*(A_k, K) \) and so no more relations are possible.

**References**