SQUARE INTEGRABLE SOLUTIONS OF $Ly=f(t, y)$

ANTON ZETTL

ABSTRACT. Let L be an ordinary linear differential operator and L^+ its formal adjoint. It is shown that, under suitable conditions on f, all solutions of $Ly=f(t, y)$ are in $L^2(0, \infty)$ provided that all those of $Ly=0$ and $L^+y=0$ are.

1. **Introduction.** For L a regular ordinary linear differential or quasi-differential operator of any order we show, under suitable conditions on f, that all solutions of

$$Ly = f(t, y) \tag{1}$$

are in $L^2(a, \infty)$, provided all solutions of

$$Ly = 0 \tag{2}$$

and

$$L^+y = 0 \tag{3}$$

are, where L^+ is the formal adjoint of L.

The case $Ly=L^+y=(ry')'+qy$ is a recent result of J. S. Bradley [4]. The linear case with $Ly=L^+y=y''+qy$ and $f(t, y)=h(t)y$ with $h(t)$ bounded is due to Bellman [3]. See also Halvorsen [6].

2. **Notation and preliminaries.** We consider first the classical operators L and L^+ defined by

$$Ly = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y,$$
$$L^+z = (\cdots(((-1)^nz' + (-1)^{n-1}p_{n-1}z)' + (-1)^{n-2}p_{n-2}z)' \cdots - p_1z)' + p_0z$$

where the p_i's for $i=0, \cdots, n-1$ are continuous real valued functions on $[a, \infty)$. This form of the adjoint equation is used to avoid smoothness assumptions on the coefficients. Denote by S and S^+ the solution spaces of $Ly=0$ and $L^+y=0$, respectively. Let y_i for $i=1, \cdots, n$ be the solutions of (2) determined by the initial conditions

$$y_i^{(j)}(a) = \delta_{i,j+1} \qquad \text{for } j = 0, \cdots, n-1.$$

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Throughout the paper δ will denote the Kronecker δ. Define $D_0^+z=z$, $D_i^+z=(D_{i-1}^+z)'+(-1)^ip_{n-i}z$ for $i=1, \cdots, n-1$.

Let z_i for $i=1, \cdots, n$ be the solutions of (3) determined by

$$D_j^+z_i(a) = (-1)^{n+i}\delta_{i,n-j} \qquad \text{for } j = 0, \cdots, n-1.$$

The proof of the main theorem will be based on two lemmas which we now state.

LEMMA 1. *Suppose $f\in C[a, \infty)$. The solution y of $Ly=f$ satisfying $y^j(a)=\alpha_{j+1}$, for $j=0, \cdots, n-1$, is given by*

$$y(t) = \sum_{i=1}^{n} \alpha_i y_i(t) + \int_a^t \sum_{i=1}^{n} y_i(t)z_i(s)f(s)ds. \tag{4}$$

This representation of solutions of nonhomogeneous equations differs from that found in most textbooks. So, although Lemma 1 follows from Theorems 2 and 3 in [11], we sketch a proof here for the sake of completeness.

PROOF. Let

$$Y' = FY \tag{5}$$

be the usual vector-matrix formulation of equation (2), i.e. Y is the column vector $[y, y', \cdots, y^{n-1}]$ and F is the matrix having all 1's in the super diagonal, $-p_0, -p_1, \cdots, -p_{n-1}$ in the bottom row from left to right and zeros elsewhere. Note that equation (3) is equivalent to the vector-matrix equation

$$Z' = F^+Z \tag{6}$$

where Z is the column vector $[D_0^+z, D_1^+z, \cdots, D_{n-1}^+z]$ and $F^+=-J^{-1}F^*J$ with $J=((-1)^i\delta_{i,\,n+1-j})$. Denote by Y_u the unique matrix solution of (5) satisfying $Y(u)=I$ and by Z_u the unique matrix solution of (6) satisfying $Z(u)=I$ where I is the $n\times n$ identity matrix.

Let $M(t, u)=Y_u(t)=(M_{ij}(t, u))$ and $N(t, u)=Z_u(t)=(N_{ij}(t, u))$ for $t\geqq a$, $u\geqq a$. Noting that $M(t, s)=M(t, u)M(u, s)$ one can readily verify that

$$M(t, u) = J^{-1}N^*(u, t)J \qquad \text{for } t \geqq a, \quad u \geqq a. \tag{7}$$

From the variation of constants formula it follows that

$$y(t) = \sum_{i=1}^{n} M_{1i}(t, a)\alpha_i + \int_a^t M_{1n}(t, u)f(u)du. \tag{8}$$

As a consequence of (8) and the definitions of M, N, y_i and z_i we can obtain that

$$M_{1n}(t, s) = \sum_{i=1}^{n} M_{1i}(t, a)M_{in}(a, s)$$
$$= \sum_{i=1}^{n} M_{1i}(t, a)(-1)^{i+n}N_{1i}(s, a) = \sum_{i=1}^{n} y_i(t)z_i(s)$$

and (4) follows.

The second lemma is a very special case of a generalization of the well-known Gronwall inequality due to H. E. Gollwitzer [**5**] (also see Willett [**9**] and Willett and Wong [**10**]).

LEMMA 2. *Let u, ϕ, g, h be nonnegative continuous functions on $[a, \infty)$ and suppose that*

$$u(t) \leqq \phi(t) + g(t)\left(\int_a^t u^2(s)h(s)ds\right)^{1/2} \qquad \text{for } t \geqq a.$$

Then

$$u(t) \leqq \phi(t) + g(t)\left\{\int_a^t 2\phi^2(s)h(s)\exp\left[\int_s^t 2g(x)h(x)dx\right]ds\right\}^{1/2}$$
$$\text{for } t \geqq a.$$

3. **The main theorem.** Suppose there exist functions h and k such that

$$|f(t, y)| \leqq k(t) + h(t)y \qquad \text{for } t \geqq a, \quad -\infty < y < \infty. \tag{9}$$

THEOREM 1. *Suppose $S\cup S^+\subset L^2(a, \infty)$ and $kz\in L^1(a, \infty)$ for all $z\in S^+$ and either*

(i) $|h|^{1/2}|y| \in L^2(a, \infty)$ *for all* $y\in S\cup S^+$ *or*

(ii) $hz\in L^2(a,\infty)$ *for all* $z\in S^+$.

Then all solutions of (1) *are in* $L^2(a,\infty)$.

PROOF. We prove first that all solutions of (1) are in $L^2(a, \infty)$ if (i) holds. Note that (9) implies all solutions are defined on $[a, \infty)$. Let y be a solution of (1).

By Lemma 1

$$y(t) = \sum_{i=1}^{n} \alpha_i y_i(t) + \int_a^t \sum_{i=1}^{n} y_i(t)z_i(s)f(s, y(s))ds.$$

Hence

$$|y(t)| \leqq \sum_{i=1}^{n} |\alpha_i| \, |y_i(t)|$$

$$+ \sum_{i=1}^{n} |y_i(t)| \int_a^t |z_i(s)| \, (|k(s)| + |h(s)y(s)|) \, ds.$$

Letting $C_i = \int_a^\infty |z_i k|$ and the Schwarz inequality we obtain

$$\begin{aligned} |y(t)| &\leqq \sum_{i=1}^{n} (C_i + |\alpha_i|) \, |y_i(t)| \\ &\quad + \sum_{i=1}^{n} |y_i(t)| \left(\int_a^t z_i^2 |h| \int_a^t y^2 |h| \right)^{1/2}. \end{aligned} \tag{10}$$

Let $D_i = (\int_a^\infty z_i^2 |h|)^{1/2}$, then $|y(t)| \leqq \phi(t) + g(t)(\int_t^a y^2 |h|)^{1/2}$ where $\phi(t) = \sum_{i=1}^n (C_i + |\alpha_i|) |y_i(t)|$ and $g(t) = \sum_{i=1}^n D_i |y_i(t)|$. It follows from Lemma 2 that

$$\begin{aligned} |y(t)| &\leqq \phi(t) + g(t) \left\{ \int_a^t 2\phi^2(s) \, |h(s)| \right. \\ &\qquad \left. \cdot \exp\left[\int_a^t 2g^2(x) \, |h(x)| \, dx \right] ds \right\}^{1/2}. \end{aligned} \tag{11}$$

Since $\int_a^\infty \phi^2(s) |h(s)| ds$ and $\int_a^\infty g^2(x) |h(x)| dx$ are both finite, we conclude that $|y|$ is bounded by a linear combination of ϕ and g and hence y is in $L^2(a, \infty)$. If (ii) holds, the proof is the same up to (10). In this case (10) becomes

$$|y(t)| \leqq \sum_{i=1}^{n} (C_i + |\alpha_i|) \, |y_i(t)| + \sum_{i=1}^{n} |y_i(t)| \left(\int_a^t z_i^2 h^2 \int_a^t y^2 \right)^{1/2}.$$

Letting $D_i = (\int_a^\infty z_i^2 h^2)^{1/2}$ and proceeding as before (11) gets replaced by

$$|y(t)| \leqq \phi(t) + g(t) \left\{ \int_a^t 2\phi^2(s) \exp\left[\int_s^t 2g(x) dx \right] ds \right\}^{1/2},$$

and the conclusion follows similarly.

4. **Remarks.** (1) Theorem 1 applied to the linear equation $Ly = hy$ yields: If $S \cup S^+ \subset L^2(a, \infty)$ and either $|h|^{1/2} y \in L^2(a, \infty)$ for all y in $S \cup S^+$ or $hz \in L^2(a, \infty)$ for all $z \in S^+$, then all solutions of $Ly = hy$ are in $L^2(a, \infty)$.

(2) The hypothesis on h is clearly satisfied if either h is bounded or $h \in L^1(a, \infty)$.

(3) Theorem 1 holds for general quasi-differential operators such as l and l^+ studied in [11]. Equivalent versions of these were considered by Shinn [8] and include, in particular, those used in [7] and [1]. See also J. H. Barrett [2] and many others. The argument given above can be readily extended to l and l^+ without any major modifications. In particular, Theorem 1 applies to operators of the type

$$Ly = (p_n y^n)^n + (p_{n-1} y^{n-1})^{n-1} + \cdots + p_0 y$$

which include the Sturm operator $(p_1 y')' + p_0 y$.

(4) Condition (9) is essentially the same as the one used by Bradley [4].

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Northern Illinois University, DeKalb, Illinois 60115