A CHARACTERIZATION OF N-COMPACT SPACES

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ABSTRACT. In this paper, we prove the following theorem:

THEOREM A. *A 0-dimensional space X is N-compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed, where N is the space of all natural numbers.*

Two consequences of Theorem A are as follows:

THEOREM B. *Suppose that X and Y are N-compact spaces. A mapping ϕ from the Boolean ring $\mathcal{B}(X)$ of all clopen subsets of X onto the Boolean ring $\mathcal{B}(X)$ of all clopen subsets of Y is an isomorphism with the property that $\bigcap_{i=1}^{\infty} A_i = \varnothing$ ($A_i \in \mathcal{B}(X)$) implies $\bigcap_{i=1}^{\infty} \phi(A_i) = \varnothing$ if and only if there exists a homeomorphism h from X onto Y such that $\phi(A) = h[A]$ for each A in $\mathcal{B}(X)$.*

THEOREM C. *A 0-dimensional space X is N-compact if and only if the collection of all the countable clopen coverings of X is complete.*

Given two spaces X and E, we say that X is *E-completely regular* (respectively *E-compact*) provided that X is homeomorphic to a subspace (respectively, closed subspace) of E^m for some cardinal number m [6]. When $E = R$, the space of all real numbers, a completely regular space X is *R-compact* (i.e. Hewitt realcompact) if and only if every z-ultrafilter on X with the countable intersection property is fixed [3, p. 77]. The purpose of this note is to prove an analogous result, namely,

THEOREM A. *A 0-dimensional space X is N-compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed, where N is the space of all natural numbers.*

A subset of a space X which is both closed and open is referred to as a *clopen* subset. $\mathcal{B}(X)$ denotes the collection of all clopen subsets of X. A *0-dimensional* space is a Hausdorff space which has a base for the topology consisting of clopen subsets.

DEFINITIONS. A clopen filter $\mathfrak{F}$ on a space X is a nonempty collection of clopen subsets of X such that

(i) $\varnothing \notin \mathfrak{F}$;

Received by the editors November 3, 1969.

AMS 1968 *subject classifications.* Primary 5453; Secondary 0660.

Key words and phrases. E-completely regular, E-compact, N-compact, clopen ultrafilter, countable intersection property, Boolean ring.

[1] A portion of this research has been done while the author was a postdoctoral fellow at the University of British Columbia. The author wishes to express his gratitude to Professor S. Mrówka for his helpful suggestions.

(ii) if $U, V \in \mathfrak{F}$ then $U \cap V \in \mathfrak{F}$; and

(iii) if $V \in \mathfrak{F}$, $U \in \mathfrak{B}(X)$ and $U \supset V$ then $U \in \mathfrak{F}$.

A clopen filter $\mathfrak{F}$ is *fixed* if $\bigcap \mathfrak{F} \neq \emptyset$ and $\mathfrak{F}$ is said to have the *countable intersection property* (abbreviated *c.i.p.*) provided that for each countable subcollection $\mathcal{G}$ of $\mathfrak{F}$, $\bigcap \mathcal{G} \neq \emptyset$. A *clopen ultrafilter* is a clopen filter which is maximal in the collection of all clopen filters ordered by inclusion.

PROOF OF THEOREM A. *Necessity*. Suppose that X is N-compact and $\mathfrak{U}$ is a clopen ultrafilter on X with $\bigcap \mathfrak{U} = \emptyset$. We shall show that $\mathfrak{U}$ does not have c.i.p.

Let $\overline{\mathfrak{U}} = \{\overline{U} = \mathrm{cl}_{\beta_{N^*}X} U : U \in \mathfrak{U}\}$, where $N^* = N \cup \{\infty\}$ is the one-point compactification of N, and $\beta_{N^*}X$ is the 0-dimensional compactification of X such that every continuous function f from X into N^* has a continuous extension f^* from $\beta_{N^*}X$ into N^*. Then $\bigcap \overline{\mathfrak{U}} \neq \emptyset$ since $\beta_{N^*}X$ is compact and $\overline{\mathfrak{U}}$ has the finite intersection property. Choose a point p in $\bigcap \overline{\mathfrak{U}}$ then $p \in \beta_{N^*}X \backslash X$. Since X is N-compact, there exists a continuous function f from X into N whose continuous extension f^* from $\beta_{N^*}X$ into N^* has the property that $f^*(p) = \infty$. Let $V_n = N^* \backslash \{1, 2, \cdots, n\}$, $n = 1, 2, 3, \cdots$. Since V_n is a clopen neighborhood of ∞, $V_n' = f^{*-1}[V_n]$ is a clopen neighborhood of p in $\beta_{N^*}X$. Since $p \in \bigcap \overline{\mathfrak{U}}$ and X is dense in $\beta_{N^*}X$, $(X \cap V_n') \cap U \neq \emptyset$ for each $U \in \mathfrak{U}$. By maximality of $\mathfrak{U}$, $X \cap V_n' \in \mathfrak{U}$ for each $n = 1, 2, \cdots$. But $\bigcap_{n=1}^{\infty} (X \cap V_n') = \emptyset$ for if $q \in \bigcap_{n=1}^{\infty} (X \cap V_n')$ then $q \in X$ and $f^*(q) = f(q) \in \bigcap_{n=1}^{\infty} V_n = \{\infty\}$. This contradicts the fact that $f[X] \subset N$. Therefore, $\mathfrak{U}$ does not have c.i.p.

Sufficiency. Assume that every clopen ultrafilter on X with c.i.p. is fixed. Consider the evaluation map $e: X \to N^{C(X,N)} = P$ where $C(X, N)$ is the collection of all continuous maps on X to N and for $x \in X$, $(ex)(f) = f(x)$ for each f in $C(X, N)$. Since X is 0-dimensional, e is a homeomorphism. Furthermore, $e[X] = \mathrm{cl}_P e[X]$, for if $p \in \mathrm{cl}_P e[X]$ and $p \notin e[X]$ then the trace on $e[X]$ of the family of all clopen neighborhoods of p in P has empty intersection. Therefore, the clopen ultrafilter $\mathfrak{U}$ on $e[X]$ containing the trace fails to have the c.i.p., say, $U_i \in \mathfrak{U}$, $i = 1, 2, \cdots$, and $\bigcap_{i=1}^{\infty} U_i = \emptyset$. Let $F_n = \bigcap_{i=1}^{n} U_i$ then $F_n \in \mathfrak{U}$, $F_n \supset F_{n+1}$ and $\bigcap_{n=1}^{\infty} F_{n=1} = \emptyset$. Define a function $f: e[X] \to N$ as follows

$$\begin{aligned} f(x) &= 1 && \text{if } x \in e[X] \backslash F_1, \\ &= n + 1 && \text{if } x \in F_n \backslash F_{n+1}. \end{aligned}$$

It is obvious that f is continuous and cannot extend continuously to the point p. This is a contradiction, since each continuous function on $e[X]$ to N, being essentially just a projection into a coordinate space, extends continuously over all of P. Thus $e[X]=\mathrm{cl}_P e[X]$, so X is N-compact.

It is well known that $\mathcal{B}(X)$ is a Boolean ring if the sum and product of two elements A, B in $\mathcal{B}(X)$ are defined by $A+B=(A\cup B)\backslash(A\cap B)$ and $AB=A\cap B$. The following theorem follows from Theorem A and we shall omit its proof since it is similar to that of [**4**, Theorem 3.2].

THEOREM B. *Suppose that X, Y are N-compact spaces. A mapping ϕ from the Boolean ring $\mathcal{B}(Y)$ of all clopen subsets of Y is an isomorphism with the property that $\bigcap_{i=1}^{\infty} A_i=\varnothing$ $(A_i\in\mathcal{B}(X))$ implies $\bigcap_{i=1}^{\infty}\phi(A_i)=\varnothing$ if and only if there exists a homeomorphism h from X onto Y such that $\phi(A)=h[A]$ for each A in $\mathcal{B}(X)$.*

According to Frolík [**2**], a space X is *almost realcompact* if $\mathcal{U}$ is an ultrafilter of open subsets of X with $\bigcap\overline{\mathcal{U}}=\varnothing$ then $\bigcap\overline{\mathcal{V}}=\varnothing$ for some countable subfamily $\mathcal{V}$ of $\mathcal{U}$. Let $\alpha=\{\mathcal{U}\}$ be a collection of coverings of a space X. An *α-Cauchy* family $\mathcal{F}$ is a filter of subsets of X such that for every $\mathcal{U}$ in α there exist A in $\mathcal{U}$ and F in $\mathcal{F}$ with $A\supset F$. The collection α will be called *complete* if $\bigcap\overline{\mathcal{F}}\neq\varnothing$ for every α-Cauchy family $\mathcal{F}$.

The following theorem is an immediate consequence of Theorem A. (Compare [**2**, Theorems 1, 13].)

THEOREM C. *A 0-dimensional space X is N-compact if and only if the collection of all clopen coverings of X is complete.*

In conclusion, we mention the following open questions.

(a) Is each 0-dimensional realcompact space necessarily N-compact?

(b) Is each 0-dimensional almost realcompact space necessarily N-compact?

(c) Is each 0-dimensional almost realcompact space necessarily realcompact?

REMARKS. It was pointed out in [**5**] that the affirmative solution of (a) was announced erroneously in [**1**]. If (b) has an affirmative solution then realcompactness, almost realcompactness and N-compactness are equivalent for 0-dimensional spaces. Finally, let $\mathcal{U}$ be a clopen ultrafilter of a 0-dimensional space X and $\mathcal{V}$ be the z-ultrafilter (respectively, ultrafilter of open subsets) containing $\mathcal{U}$. Then it follows from Theorem A that a sufficient condition for (a) (respectively, (b)) to have an affirmative solution is the statement that $\mathcal{V}$ (respectively, $\overline{\mathcal{V}}$) has the c.i.p. if $\mathcal{U}$ does.

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