A CHARACTERIZATION OF N-COMPACT SPACES

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Abstract. In this paper, we prove the following theorem:

Theorem A. A 0-dimensional space X is N-compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed, where N is the space of all natural numbers.

Two consequences of Theorem A are as follows:

Theorem B. Suppose that X and Y are N-compact spaces. A mapping $\phi$ from the Boolean ring $\mathcal{B}(X)$ of all clopen subsets of X onto the Boolean ring $\mathcal{B}(Y)$ of all clopen subsets of Y is an isomorphism with the property that $\bigcap_{i=1}^{n} A_i = \emptyset$ if and only if there exists a homeomorphism $h$ from X onto Y such that $\phi(A) = h[A]$ for each $A$ in $\mathcal{B}(X)$.

Theorem C. A 0-dimensional space X is N-compact if and only if the collection of all the countable clopen coverings of X is complete.

Given two spaces X and E, we say that X is E-completely regular (respectively E-compact) provided that X is homeomorphic to a subspace (respectively, closed subspace) of $E^m$ for some cardinal number $m$ [6]. When $E = R$, the space of all real numbers, a completely regular space X is R-compact (i.e. Hewitt realcompact) if and only if every $z$-ultrafilter on X with the countable intersection property is fixed [3, p. 77]. The purpose of this note is to prove an analogous result, namely,

Theorem A. A 0-dimensional space X is N-compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed, where N is the space of all natural numbers.

A subset of a space X which is both closed and open is referred to as a clopen subset. $\mathcal{B}(X)$ denotes the collection of all clopen subsets of X. A 0-dimensional space is a Hausdorff space which has a base for the topology consisting of clopen subsets.

Definitions. A clopen filter $\mathcal{F}$ on a space X is a nonempty collection of clopen subsets of X such that

(i) $\emptyset \in \mathcal{F}$;

Received by the editors November 3, 1969.

AMS 1968 subject classifications. Primary 5453; Secondary 0660.

Key words and phrases. E-completely regular, E-compact, N-compact, clopen ultrafilter, countable intersection property, Boolean ring.

1 A portion of this research has been done while the author was a postdoctoral fellow at the University of British Columbia. The author wishes to express his gratitude to Professor S. Mrówka for his helpful suggestions.
(ii) if \( U, V \in \mathcal{F} \) then \( U \cap V \in \mathcal{F} \); and

(iii) if \( V \in \mathcal{F} \), \( U \in \mathcal{B}(X) \) and \( U \supset V \) then \( U \in \mathcal{F} \).

A clopen filter \( \mathcal{F} \) is fixed if \( \bigcap \mathcal{F} \neq \emptyset \) and \( \mathcal{F} \) is said to have the countable intersection property (abbreviated c.i.p.) provided that for each countable subcollection \( \mathcal{G} \) of \( \mathcal{F} \), \( \bigcap \mathcal{G} \neq \emptyset \). A clopen ultrafilter is a clopen filter which is maximal in the collection of all clopen filters ordered by inclusion.

**Proof of Theorem A. Necessity.** Suppose that \( X \) is \( N \)-compact and \( \mathcal{U} \) is a clopen ultrafilter on \( X \) with \( \bigcap \mathcal{U} = \emptyset \). We shall show that \( \mathcal{U} \) does not have c.i.p.

Let \( \mathcal{U} = \{ \overline{U} = \text{cl}_{\beta N \mathcal{X}} U : U \in \mathcal{U} \} \), where \( N^* = N \cup \{ \infty \} \) is the one-point compactification of \( N \), and \( \beta N \mathcal{X} \) is the 0-dimensional compactification of \( X \) such that every continuous function \( f \) from \( X \) into \( N^* \) has a continuous extension \( f^* \) from \( \beta N \mathcal{X} \) into \( N^* \). Then \( \mathfrak{N} \neq \emptyset \) since \( \beta N \mathcal{X} \) is compact and \( \mathfrak{N} \) has the finite intersection property. Choose a point \( p \) in \( \mathfrak{N} \) then \( p \in \beta N \mathcal{X} \backslash X \). Since \( X \) is \( N \)-compact, there exists a continuous function \( f \) from \( X \) into \( N \) whose continuous extension \( f^* \) from \( \beta N \mathcal{X} \) into \( N^* \) has the property that \( f^*(p) = \infty \).

Let \( V_n = N^* \setminus \{ 1, 2, \ldots, n \} \), \( n = 1, 2, 3, \ldots \). Since \( V_n \) is a clopen neighborhood of \( \infty \), \( V_n = f^{n-1} \{ V_n \} \) is a clopen neighborhood of \( p \) in \( \beta N \mathcal{X} \). Since \( p \in \mathfrak{N} \) and \( X \) is dense in \( \beta N \mathcal{X} \), \( (X \cap V_n) \cap U \neq \emptyset \) for each \( U \in \mathcal{U} \). By maximality of \( \mathcal{U} \), \( X \cap V_n \in \mathcal{U} \) for each \( n = 1, 2, \ldots \).

But \( \bigcap_{n=1}^\infty (X \cap V_n) = \emptyset \) for if \( q \in \bigcap_{n=1}^\infty (X \cap V_n) \) then \( q \in X \) and \( f^*(q) = f(q) \in \bigcap_{n=1}^\infty V_n = \{ \infty \} \). This contradicts the fact that \( f[X] \subset N \). Therefore, \( \mathcal{U} \) does not have c.i.p.

**Sufficiency.** Assume that every clopen ultrafilter on \( X \) with c.i.p. is fixed. Consider the evaluation map \( e : X \rightarrow N^{C(X, N)} = P \) where \( C(X, N) \) is the collection of all continuous maps on \( X \) to \( N \) and for \( x \in X \), \( (ex)(f) = f(x) \) for each \( f \) in \( C(X, N) \). Since \( X \) is 0-dimensional, \( e \) is a homeomorphism. Furthermore, \( e[X] = \text{cl}_P e[X] \), for if \( p \in \text{cl}_P e[X] \) and \( p \in e[X] \) then the trace on \( e[X] \) of the family of all clopen neighborhoods of \( p \) in \( P \) has empty intersection. Therefore, the clopen ultrafilter \( \mathfrak{U} \) on \( e[X] \) containing the trace fails to have the c.i.p., say, \( U_i \in \mathfrak{U} \), \( i = 1, 2, \ldots \), and \( \bigcap_{i=1}^\infty U_i = \emptyset \). Let \( F_n = \bigcap_{i=1}^n U_i \), then \( F_n \in \mathfrak{U} \), \( F_n \supset F_{n+1} \) and \( \bigcap_{i=1}^\infty F_{n+1} = \emptyset \). Define a function \( f : e[X] \rightarrow N \) as follows

\[
\begin{align*}
f(x) &= 1 \quad \text{if } x \in e[X] \setminus F_1, \\
&= n + 1 \quad \text{if } x \in F_n \setminus F_{n+1}.
\end{align*}
\]

It is obvious that \( f \) is continuous and cannot extend continuously to the point \( p \). This is a contradiction, since each continuous function...
on $e[X]$ to $N$, being essentially just a projection into a coordinate space, extends continuously over all of $P$. Thus $e[X] = \text{cl}_P e[X]$, so $X$ is $N$-compact.

It is well known that $\mathfrak{B}(X)$ is a Boolean ring if the sum and product of two elements $A, B$ in $\mathfrak{B}(X)$ are defined by $A + B = (A \cup B) \setminus (A \cap B)$ and $AB = A \cap B$. The following theorem follows from Theorem A and we shall omit its proof since it is similar to that of [4, Theorem 3.2].

**Theorem B.** Suppose that $X$, $Y$ are $N$-compact spaces. A mapping $\phi$ from the Boolean ring $\mathfrak{B}(Y)$ of all clopen subsets of $Y$ is an isomorphism with the property that $\bigcap_{i=1}^{n} A_i = \emptyset$ $(A_i \subseteq \mathfrak{B}(X))$ implies $\bigcap_{i=1}^{n} \phi(A_i) = \emptyset$ if and only if there exists a homeomorphism $h$ from $X$ onto $Y$ such that $\phi(A) = h[A]$ for each $A$ in $\mathfrak{B}(X)$.

According to Frolik [2], a space $X$ is almost realcompact if every ultrafilter of open subsets of $X$ with $\bigcap_{i} \mathcal{U} = \emptyset$ then $\bigcap_{i} \mathcal{V} = \emptyset$ for some countable subfamily $\mathcal{V}$ of $\mathcal{U}$. Let $\alpha = \{\mathcal{U}\}$ be a collection of coverings of a space $X$. An $\alpha$-Cauchy family $\mathcal{F}$ is a filter of subsets of $X$ such that for every $\mathcal{U}$ in $\alpha$ there exist $A$ in $\mathcal{U}$ and $F$ in $\mathcal{F}$ with $A \supseteq F$. The collection $\alpha$ will be called complete if $\bigcap_{i} \mathcal{F} \neq \emptyset$ for every $\alpha$-Cauchy family $\mathcal{F}$.

The following theorem is an immediate consequence of Theorem A. (Compare [2, Theorems 1, 13].)

**Theorem C.** A 0-dimensional space $X$ is $N$-compact if and only if the collection of all clopen coverings of $X$ is complete.

In conclusion, we mention the following open questions.

(a) Is each 0-dimensional realcompact space necessarily $N$-compact?

(b) Is each 0-dimensional almost realcompact space necessarily $N$-compact?

(c) Is each 0-dimensional almost realcompact space necessarily realcompact?

**Remarks.** It was pointed out in [5] that the affirmative solution of (a) was announced erroneously in [1]. If (b) has an affirmative solution then realcompactness, almost realcompactness and $N$-compactness are equivalent for 0-dimensional spaces. Finally, let $\mathcal{U}$ be a clopen ultrafilter of a 0-dimensional space $X$ and $\mathcal{V}$ be the $\varepsilon$-ultrafilter (respectively, ultrafilter of open subsets) containing $\mathcal{U}$. Then it follows from Theorem A that a sufficient condition for (a) (respectively, (b)) to have an affirmative solution is the statement that $\mathcal{V}$ (respectively, $\mathcal{V}$) has the c.i.p. if $\mathcal{U}$ does.
References


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