REGULAR-CLOSED, URYSOHN-CLOSED AND COMPLETELY HAUSDORFF-CLOSED SPACES

H. HERRLICH

Abstract. Recently M. P. Berri, J. R. Porter, and R. M. Stephenson, Jr. have given a survey on $P$-closed and $P$-minimal spaces. In the present paper the first two problems of this survey will be solved: (1) The product of two Urysohn-closed spaces need not be Urysohn-closed. (2) A completely Hausdorff-closed regular space need not be regular-closed.

Given a topological property $P$ a $P$-space $X$ is called $P$-closed provided $X$ is a closed set in every $P$-space in which it can be embedded. For instance compact Hausdorff spaces are Hausdorff-closed but not vice versa. $P$-closed spaces (and the closely related $P$-minimal spaces) have been investigated extensively especially for separation properties $P = $ Hausdorff, Urysohn, regular (includes $T_1$). One of the major results states that $P$-closure (as well as $P$-minimality) is productive for $P = $ Hausdorff. The corresponding problem for $P = $ Urysohn and $P = $ regular has been attacked by several authors. Scarborough and Stephenson proved independently that the product of an Urysohn-closed space and a Hausdorff-closed Urysohn space is Urysohn-closed. In §1 of the present paper we shall show that the product of two Urysohn-closed spaces need not be Urysohn-closed (Theorem 1). This settles Problem 2 of the survey of $P$-closed and $P$-minimal spaces presented recently by M. P. Berri, J. R. Porter and R. M. Stephenson, Jr. [2]. In addition in §2 we shall give a negative answer to Problem 1 of the above mentioned survey: Is a completely Hausdorff-closed regular space necessarily regular-closed? (Theorem 2.)

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1. Products of Urysohn-closed spaces. A topological space $X$ is called a Urysohn-space provided that any two points of $X$ have disjoint closed neighborhoods. It is easy to show that a Urysohn-space $X$ is Urysohn-closed iff for any open filter $F$ in $X$ there exists a point
Theorem 1. There exist Urysohn-closed spaces $Y$ and $Z$ whose product $X = Z \times Y$ is not Urysohn-closed.

Proof. (a) Construction of $Y$. Let $W$ be the set of all countable ordinals, and let $I = [0, 1)$ be the half-open unit interval of the reals, both sets supplied with the usual order. The product $P = W \times I$ can be ordered lexicographically and be supplied with the corresponding order topology. Let $A$ denote the resulting space (Alexandroff's long line) and let $A' = A \cup \{a\}$ be the one-point-compactification of $A$. If $(D_1, D_2, D_3)$ is a fixed partition of $I$ into three pairwise disjoint dense subsets (with $0 \in D_2$) then $(E_1 = W \times D_1, E_2 = W \times D_2, E_3 = (W \times D_3) \cup \{a\})$ is a partition of $A'$ into three pairwise disjoint dense subsets. If $B$ is the topology (= family of open sets) of $A'$ then $B \cup \{E_2, E_3\}$ is a subbase for a new topology of $A'$. The resulting space $Y$ is Urysohn-closed.

(b) Construction of $Z$. For $i = 0, 1$ let $w_i$ denote the least ordinal with cardinality $\aleph_i$ and let $W_i$ denote the set of all ordinals $\alpha$ with $\alpha \leq w_i$ supplied with the usual order and the corresponding order topology. If we remove the point $(w_i, w_0)$ from the product space $W_i \times W_0$ we obtain the well-known Tychonoff plank $T$. Let $T_1$ and $T_2$ be two copies of $T$ whose elements will be denoted by $(\alpha, n, 1)$ and $(\alpha, n, 2)$ respectively. For fixed $\alpha < w_1, n < w_0, i \in \{0, 1\}$, define $U_i(\alpha, n) = \{\beta, m, i\} | \alpha < \beta < w_1, n < m \leq w_0\}$ and $V_i(\alpha, n) = \{\beta, m, i\} | \beta < \alpha, m < n\}$. In the topological union of $T_1$ and $T_2$ we identify for any $n < w_0$ the two points $(w_i, n, 1)$ and $(w_i, n, 2)$. To the resulting space $Q$ we add a point $t$ and define the set $\{U_i(\alpha, n) \cup \{t\} | \alpha < w_1, n < w_0\}$ to be a base for the neighborhoods of $t$. The resulting space $Z$ is Urysohn-closed.

(c) $X = Z \times Y$ is not Urysohn-closed. $F_0 = \{(\alpha, n, 2), (\gamma, r)\} | (\alpha, n, 2) \in T_2, \alpha < \gamma < w_1, r \in D_2\}$ is an open subset of $X$, since for any point $p = ((\alpha, n, 2), (\gamma, r))$ of $F_0$ we have $p \in V_2(\alpha + 1, n + 1) \times \{y\} | y \in E_2, (\gamma, r/2) < y\} \subset F_0$. Consequently for any $\alpha < w_1, n < w_0, F(\alpha, n) = F_0 \cap (U_2(\alpha, n) \times Y)$ is open in $X$ and $\{F(\alpha, n) | \alpha < w_1, n < w_0\}$ is a base for an open filter $\mathfrak{F}$ on $X$. It remains to show that for each point $x$ of $X$ there exists a member $F$ of $\mathfrak{F}$ and an open set $U$ containing a closed neighborhood of $x$ with $F \cap U = \emptyset$. This is easy to see for $x \not= (t, a)$. In case $x = (t, a)$, the set $V = (T_1 \cup \{t\}) \times (E_1 \cup E_2)$ is a closed neighborhood of $x$. It is sufficient to show that any point $V$ has a neighborhood which does not meet $F_0$. This is obvious for $\varepsilon \in \text{Cl}_2T_2$ or
y \in \text{Cl}_Y E_z$. Consequently we can assume $z = (w_1, n, 1) = (w_1, n, 2)$ and $y = (\alpha, r)$ with $r \in D_r$. But then $W = \{(\beta, n, i) | \alpha + 1 < \beta \leq w_1, i \in \{1, 2\}\} \times \{y' | y' \in Y, y < (\alpha + 1, 0)\}$ is a neighborhood of $(z, y)$ which does not meet $F_0$ since $(\lambda, m, i), (\xi, s) \in W \cap F_0$ would imply $\lambda < \xi < \alpha + 1 < \lambda$, which is impossible.

2. Regular-closed and completely Hausdorff-closed spaces. A topological space $X$ is called completely Hausdorff provided that for any two distinct points $x$ and $y$ of $X$ there exists a continuous, real-valued function $f : X \to \mathbb{R}$ with $f(x) \neq f(y)$ (equivalently: provided there exists a continuous one-to-one map of $X$ into a compact Hausdorff space).

Theorem 2. There exists a completely Hausdorff-closed regular space $X$ which is not regular-closed.

Proof. Let $T$ be the Tychonoff plank described in §1, let $Z$ be the set of integers, and let $R$ be the topological union of countably many copies $T_i$ of $T$ ($i \in \mathbb{Z}$) whose elements will be denoted respectively by $(\alpha, n, i)$. Identify in $R$ for any $i \in \mathbb{Z}$ and any $\alpha < w_1$ the points $(\alpha, w_0, 2i)$ and $(\alpha, w_0, 2i + 1)$, and for any $i \in \mathbb{Z}$ and any $n < w_0$ the points $(w_1, n, 2i - 1)$ and $(w_1, n, 2i)$. Let $Q$ be the corresponding quotient space of $R$ and let $\beta Q$ be its Čech-Stone-compactification. Then there exists exactly one point $q$ in $\beta Q$ such that each neighborhood of $q$ meets each $T_i$. Form a new space $P$ by replacing the point $q$ in $\beta Q$ by two different points $q^+$ and $q^-$ and calling a subset $U$ of $P$ a neighborhood of $q^+$ (resp. $q^-$) in $P$ iff $U$ contains $q^+$ (resp. $q^-$) and there exists an $i \in \mathbb{Z}$ such that $(U \setminus \{q^+\}) \cup \text{Cl}_{\beta Q}(U_{j, i, T_j})$ (resp. $(U \setminus \{q^+\}) \cup \text{Cl}_{\beta Q}(U_{j, i, T_j})$) is a neighborhood of $q$ in $\beta Q$. It is easy to verify that $P$ is regular using the fact that for $n \in \mathbb{Z}$, $\text{Cl}_{\beta Q}(U_{j > n, T_j}) \cap \text{Cl}_{\beta Q}(U_{j < n, T_j}) = \{q\}$; the proof of this fact is straightforward. Consequently, the subspace $X$ obtained from $P$ by removing $q^-$ is not regular-closed. But $X$ is completely Hausdorff-closed. For let $Y$ be a Hausdorff space which contains $X$ as a nonclosed subspace. Then there exists a point $y \in \text{Cl}_Y X - X$. Each $Y$-neighborhood of $y$ meets each $\beta Q$-neighborhood of $q$ in $Q$. Consequently there exists no continuous function $F : Y \to \mathbb{R}$ with $F(y) \neq F(q^+)$). Hence $Y$ is not completely Hausdorff.

References


\textsc{University of Florida, Gainesville, Florida 32601}