ON THE OPERATIONAL CALCULUS FOR GROUPS OF OPERATORS[1]

SHMUEL KANTOROVITZ

ABSTRACT. We study some compactness properties of the natural operational calculus for groups of operators.

1. Let $\{T(t) \mid t \text{ real}\}$ be a strongly continuous group of operators on a Banach space X. Let M_T be the class of all complex Borel measures m on the real line R such that

$$\|m\|_T = \int \|T(t)\| d\,|m|\,(t) < \infty.$$

M_T is a Banach algebra for the usual operations with measures and for the norm $\|\cdot\|_T$. The image A_T of M_T under the Fourier transform $m \to \hat{m}$ is a Banach algebra of bounded continuous complex functions on R, with the usual pointwise operations and with the norm $\|\hat{m}\|_T = \|m\|_T$.

If m_t denotes the delta measure at t and $f_t(s) = \hat{m}_t(s) = \exp(its)$ $(t, s \in R)$, then $m_t \in M_T$ $(f_t \in A_T)$ and

$$\|f_t\|_T = \|m_t\|_T = \|T(t)\|, \qquad t \in R. \tag{1}$$

Let F be a Banach algebra of complex functions on R which contains the functions f_t $(t \in R)$. An F-operational calculus for $T(\cdot)$ is a continuous representation of F on X which sends f_t to $T(t)$ for all $t \in R$.

For $f \in A_T$, define the operator $\tau(f)$ on X by

$$\tau(f)x = \int T(t)x\,dm(t) \qquad (x \in X,\ f = \hat{m} \in A_T). \tag{2}$$

τ is an A_T-operational calculus for $T(\cdot)$ with $\|\tau\| = 1$ (cf. (1)); we call it *the natural operational calculus for* $T(\cdot)$.

The operational calculus may be localized as follows. For $0 \neq x \in X$,

Received by the editors March 3, 1970.

AMS 1969 *subject classifications*. Primary 4730; Secondary 4750.

Key words and phrases. Group of operators, operational calculus, compact operator, compactly imbedded, regular vectors, spectral operator, quasi-nilpotent operator.

[1] Research supported by NSF Grant GP 8289.

let A_{Tx} denote the Banach space of all Fourier transforms of complex Borel measures m on R for which

$$\|\hat{m}\|_{Tx} = \int \|T(t)x\| d\,|\,m\,|\,(t) < \infty,$$

with the norm $\|\cdot\|_{Tx}$. The map $\tau(\cdot)x: A_{Tx} \to X$ defined by (2) is a continuous linear mapping of norm 1, such that $\tau(f_t)x = T(t)x$ and $\tau(fg)x = \tau(f)\tau(g)x$ $(f \in A_T,\ g \in A_{Tx})$.

DEFINITION 1. $x \in X$ is (*strongly*) *regular* for $T(\cdot)$ if the ratio $\|T(\cdot)x\|/\|T(\cdot)\|$ has a positive (infimum) lim sup as $|t| \to \infty$.

THEOREM 1. *Let $S(\cdot)$ and $T(\cdot)$ be strongly continuous groups of operators, and let $x \in X$. Then*

(i) *$A_S \subset A_T$ iff $\|T(\cdot)\|/\|S(\cdot)\|$ is bounded.*

(ii) *$A_{Tx} = A_T$ iff x is strongly regular for $T(\cdot)$.*

PROOF. Let C_0' be the dual of the space C_0 of all complex continuous functions with compact support on R (with the usual topology). If $h \in C_0$, $Q = \operatorname{supp} h$, and $K = \sup_Q \|T(\cdot)\|^{-1}$ (finite!), then

$$\left| \int h dm \right| \leqq K \sup |\,h\,| \cdot \|m\|_T,$$

so that M_T is topologically contained in C_0'. Now if $A_S \subset A_T$, then $M_S \subset M_T$, and the latter (hence the former) inclusion is topological, since both spaces are Banach spaces topologically contained in C_0' (cf. the Closed Graph Theorem). Since $f_t \in A_S$, it follows that

$$\|T(t)\| = \|f_t\|_T \leqq B\|f_t\|_S = B\|S(t)\|, \qquad t \in R.$$

This proves the nontrivial part of (i), and (ii) is proved by the same argument.

2. The group $T(\cdot)$ is *bounded away from zero* if $\inf\|T(\cdot)\| > 0$, and *temperate* if $\|T(t)\| = O(|t|^k)$ as $|t| \to \infty$, for some nonnegative integer k. Let

$$w^{\pm} = \lim_{t \to \pm\infty} t^{-1} \log\|T(t)\|$$

(cf. [2, p. 244]). The spectral norm of $T(t)$ is $\exp[tw^+]$ for $t \geqq 0$ and $\exp[tw^-]$ for $t < 0$; this implies the equivalence of the following statements:

1. $T(\cdot)$ is bounded away from zero;
2. $\|T(\cdot)\| \geqq 1$;
3. $w^- \leqq 0 \leqq w^+$.

In particular, *temperate groups are bounded away from* 0.

For $Q \subset R$ compact, denote by $A_T(Q)$ the closed subspace of A_T consisting of all $f \in A_T$ with support in Q. If $f = \hat{m} \in A_T(Q)$, then $dm(t) = \hat{f}(-t)dt/2\pi$, and therefore

$$\|f\|_T = \int \|T(t)\| \, |\hat{f}(-t)| \, dt/2\pi, \qquad f \in A_T(Q). \tag{3}$$

THEOREM 2. *Let $S(\cdot)$ and $T(\cdot)$ be strongly continuous groups of operators. If $S(\cdot)$ is bounded away from* 0 *and $\|T(t)\|/\|S(t)\| \to 0$ as $|t| \to \infty$, then, for each compact $Q \subset R$ and $x \in X$, $A_S(Q)$ is compactly imbedded in A_T and the restrictions $\tau | A_S(Q)$ and $\tau(\cdot)x | A_S(Q)$ are compact operators. Conversely, if $S(\cdot)$ and $T(\cdot)$ are temperate and $A_S(Q)$ is compactly imbedded in A_T for some compact Q with nonvoid interior, then $\|T(t)\|/\|S(t)\| \to 0$ as $|t| \to \infty$.*

PROOF. 1. Let $S(\cdot)$ be bounded away from 0 (hence $\|S(\cdot)\| \geqq 1$), and let $\|T(t)\|/\|S(t)\| \to 0$ as $|t| \to \infty$. Given $\epsilon > 0$, choose $N > 0$ such that $\|T(t)\|/\|S(t)\| < \epsilon$ for $|t| > N$. Let $Q \subset R$ be compact, and let $\{h_n'\}$ be in the closed unit ball of $A_S(Q)$. By (3), $\{\hat{h}_n'\}$ is in the closed 2π-ball of $L^1(R)$. Therefore, a subsequence $\{\hat{h}_n\}$ converges weak* in $(L^\infty)^*$. In particular, for each $s \in R$

$$\frac{1}{2\pi} \int e^{-its} \hat{h}_n(t) dt = h_n(s)$$

converge. Since

$$|h_n(s)| \leqq \frac{1}{2\pi} \|\hat{h}_n\|_{L^1} \leqq 1 \quad \text{and} \quad \hat{h}_n(t) = \int_Q e^{its} h_n(s) ds,$$

it follows by Dominated Convergence that $\hat{h}_n$ converge pointwise on R and $|\hat{h}_n| \leqq |Q|$ (the Lebesgue measure of Q). Now, by (3),

$$\|h_n - h_m\|_T \leqq \int_{-N}^{N} \|T(t)\| \, |\hat{h}_n(-t) - \hat{h}_m(-t)| \, dt/2\pi + \epsilon \|h_n - h_m\|_S.$$

The first term on the right tends to 0 as $n, m \to \infty$ (Dominated Convergence), and the second term is $\leqq 2\epsilon$. Hence $\{h_n\}$ is Cauchy in A_T, and the first part of the theorem follows.

2. Let $S(\cdot)$ and $T(\cdot)$ be temperate, and suppose $A_S(Q)$ is compactly imbedded in A_T for some compact Q with nonvoid interior. There exists a nonzero C^∞ function g with support in Q. Let $g_t = gf_t/\|S(t)\|$ $(t \in R)$. Then $g_t \in A_S(Q)$, and by (3),

$$\|g_t\|_S = \int \|S(s)\| \, |\hat{g}(t-s)| \, \|S(t)\|^{-1} ds/2\pi$$

$$= \int \|S(t+u)\| \, \|S(t)\|^{-1} \, |\hat{g}(-u)| \, du/2\pi$$

$$\leqq \int \|S(u)\| \, |\hat{g}(-u)| \, du/2\pi = \|g\|_S \qquad (t \in R).$$

Therefore, the set $\{g_t | t \in R\}$ has limit points in A_T, which are also limit points in the topology induced on A_T by the weak* topology on $L^\infty(R)$ (because $\|T(\cdot)\| \geqq 1$). However, $g_t \to 0$ (as $|t| \to \infty$) in the latter topology, since for each $h \in L^1(R)$,

$$\left| \int h(s) g_t(s) ds \right| = |(hg)\hat{}(t)| / \|S(t)\| \leqq |(hg)\hat{}(t)| \to 0$$

by the Riemann-Lebesgue lemma. Therefore $g_t \to 0$ in A_T as $|t| \to \infty$. Now

$$\|g_t\|_T = \int \|T(s)\| \, |\hat{g}(t-s)| \, ds/(2\pi \|S(t)\|)$$

$$\geqq (\|T(t)\|/\|S(t)\|) \int \|T(t-s)\|^{-1} |\hat{g}(t-s)| \, ds/2\pi$$

$$= c\|T(t)\|/\|S(t)\|,$$

where $c = \int \|T(u)\|^{-1} |\hat{g}(u)| du/2\pi$ is a positive constant. Since $\|g_t\|_T \to 0$, the result follows.

The same arguments (with the estimate $\|T(s)x\| \geqq \|T(t)x\| \cdot \|T(t-s)\|^{-1}$ replacing the inequality $\|T(s)\| \geqq \|T(t)\| \, \|T(t-s)\|^{-1}$ in the last part of the proof) prove the following.

THEOREM 3. *Let $T(\cdot)$ be a strongly continuous group of operators, and let $x \in X$. If $T(\cdot)$ is bounded away from zero and x is nonregular, then $A_T(Q)$ is compactly imbedded in A_{Tx} for each compact $Q \subset R$. Conversely, if $T(\cdot)$ is temperate and $A_T(Q)$ is compactly imbedded in A_{Tx} for some compact set Q with nonvoid interior, then x is nonregular.*

COROLLARY. *Let $T(\cdot)$ be a temperate strongly continuous group of operators. Then x is a regular vector for $T(\cdot)$ iff the injection $A_T(Q) \to A_{Tx}$ is not compact for any compact set Q with nonvoid interior.*

3. **Remarks.** The existence of regular vectors seems to be a difficult question in general. Observe that if $S(\cdot)$ and $N(\cdot)$ are commuting

groups of operators, and $S(\cdot)$ is uniformly bounded, then the groups $T(\cdot)=S(\cdot)N(\cdot)$ and $N(\cdot)$ have the same (strongly) regular vectors. This follows from the estimate (with $K=\sup_R\|S(\cdot)\|$):

$$\|T(t)x\|/\|T(t)\| \geqq K^{-1}\|S(t)N(t)x\|/\|N(t)\| \geqq K^{-2}\|N(t)x\|/\|N(t)\|.$$

In particular, every nonzero vector is strongly regular for a uniformly bounded group.

PROPOSITION. *If T is a real spectral operator of finite type, then the group $T(t)=e^{itT}$ possesses regular vectors.*

PROOF. Let $T=S+N$ be the canonical decomposition of T (cf. [1]). Then $T(\cdot)=S(\cdot)N(\cdot)$, where $S(t)=e^{itS}$ and $N(t)=e^{itN}$ are commuting groups and $S(\cdot)$ is uniformly bounded. By the preceding remark, it suffices to show that $N(\cdot)$ possesses regular vectors. Let k be the nonnegative integer defined by the requirements $N^k\neq 0$ and $N^{k+1}=0$. Then

$$(it)^{-k}N(t) = \sum_{j=0}^{k-1}(it)^{j-k}N^j/j! + N^k/k! \to N^k/k!$$

as $|t|\to\infty$, in the uniform operator topology. Therefore,

$$\lim_{|t|\to\infty}\|N(t)x\|/\|N(t)\| = \lim_{|t|\to\infty}\|(it)^{-k}N(t)x\|/\|(it)^{-k}N(t)\|$$
$$= \|N^kx\|/\|N^k\|,$$

so that each vector not in the null space of N^k is regular for $N(\cdot)$.

CONJECTURE. Every temperate group of operators possesses regular vectors.

The following example of R. Beals shows that the "finite type" hypothesis cannot be omitted in the above proposition.

For $n=1, 2, \cdots$, let N_n be the $n\times n$ matrix with $1/n$ in the first superdiagonal and 0 elsewhere, and consider $N=\sum\oplus N_n$ as an operator in l_1. Since $N_n^n=0$ and $\|N_n^k\|=n^{-k}$ for $k<n$, we have $N^k=\sum_{n>k}\oplus N_n^k$ and $\|N^k\|=\sup_{n>k}n^{-k}=(k+1)^{-k}$. Hence N is quasi-nilpotent. Let $N(t)=e^{tN}=\sum\oplus\exp(tN_n)$. One calculates that $\|\exp(tN_n)\|=\sum_{k<n}(t/n)^k/k!$. Since

$$\|N(t)\| = \sup_n\|\exp(tN_n)\|,$$

it follows that for each fixed n,

$$\|\exp(tN_n)\|/\|N(t)\| \leqq \|\exp(tN_n)\|/\|\exp(tN_{n+1})\| \to 0 \tag{1}$$

as $t\to\infty$.

Let $P_n:(x_1, x_2, \cdots)\to(x_1, \cdots, x_{m(n)}, 0, 0, \cdots)$, where $m(n) = 1+\cdots+n$. For any $x\in l_1$ and $n=1, 2, \cdots$,

$$\begin{aligned}\|N(t)x\|/\|N(t)\| &\leqq \|N(t)P_nx\|/\|N(t)\| + \|N(t)(I-P_n)x\|/\|N(t)\| \\ &\leqq \|\exp(tN_n)\|\,\|x\|/\|N(t)\| + \|(I-P_n)x\|,\end{aligned}$$

and therefore,

$$\limsup_{t\to\infty}\|N(t)x\|/\|N(t)\| \leqq \|(I-P_n)x\|,$$

by (1). Letting $n\to\infty$, we obtain that the lim sup is 0. Similarly, the lim sup as $t\to-\infty$ is 0. Since x was arbitrary, this shows that $N(\cdot)$ has no regular vectors.

References

1. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64** (1958), 217–274. MR **21** #3616.

2. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957. MR **19**, 664.

University of Illinois at Chicago Circle, Chicago, Illinois 60680