ON THE OPERATIONAL CALCULUS FOR GROUPS OF OPERATORS

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Abstract. We study some compactness properties of the natural operational calculus for groups of operators.

1. Let \( \{ T(t) \mid t \text{ real} \} \) be a strongly continuous group of operators on a Banach space \( X \). Let \( M_T \) be the class of all complex Borel measures \( m \) on the real line \( R \) such that

\[
\| m \|_T = \int \| T(t) \| \, d \| m \| \leq \infty.
\]

\( M_T \) is a Banach algebra for the usual operations with measures and for the norm \( \| m \|_T \). The image \( A_T \) of \( M_T \) under the Fourier transform \( m \to \hat{m} \) is a Banach algebra of bounded continuous complex functions on \( R \), with the usual pointwise operations and with the norm \( \| \hat{m} \|_T = \| m \|_T \).

If \( m_t \) denotes the delta measure at \( t \) and \( f_t(s) = m_t(s) = \exp(its) \) \((t, s \in R)\), then \( m_t \in M_T \) \( (f_t \in A_T) \) and

\[
\| f_t \|_T = \| m_t \|_T = \| T(t) \|, \quad t \in R.
\]

Let \( F \) be a Banach algebra of complex functions on \( R \) which contains the functions \( f_t \) \((t \in R)\). An \( F \)-operational calculus for \( T(\cdot) \) is a continuous representation of \( F \) on \( X \) which sends \( f_t \) to \( T(t) \) for all \( t \in R \).

For \( f \in A_T \), define the operator \( \tau(f) \) on \( X \) by

\[
\tau(f)x = \int T(t)x \, d\hat{m}(t) \quad (x \in X, f = \hat{m} \in A_T).
\]

\( \tau \) is an \( A_T \)-operational calculus for \( T(\cdot) \) with \( \| \tau \| = 1 \) (cf. (1)); we call it the natural operational calculus for \( T(\cdot) \).

The operational calculus may be localized as follows. For \( 0 \neq x \in X \),
let $A_{T_x}$ denote the Banach space of all Fourier transforms of complex Borel measures $m$ on $R$ for which
\[ \|m\|_{T_x} = \int \|T(t)x\|d\ |m| (t) < \infty, \]
with the norm $\| \cdot \|_{T_x}$. The map $\tau(\cdot)x: A_{T_x} \to X$ defined by (2) is a continuous linear mapping of norm 1, such that $\tau(f_i)x = T(t)x$ and $\tau(fg)x = \tau(f)\tau(g)x$ $(f \in A_T, g \in A_{T_x})$.

**Definition 1.** $x \in X$ is strongly regular for $T(\cdot)$ if the ratio $\|T(\cdot)x\|/\|T(\cdot)\|$ has a positive (infimum) lim sup as $|t| \to \infty$.

**Theorem 1.** Let $S(\cdot)$ and $T(\cdot)$ be strongly continuous groups of operators, and let $x \in X$. Then
1. $A_S \subseteq A_T$ iff $\|T(\cdot)\|/\|S(\cdot)\|$ is bounded.
2. $A_{T_x} = A_T$ iff $x$ is strongly regular for $T(\cdot)$.

**Proof.** Let $C_0'$ be the dual of the space $C_0$ of all complex continuous functions with compact support on $R$ (with the usual topology). If $h \in C_0$, $Q = \text{supp } h$, and $A'_s = \sup_{q} \|T(\cdot)\|^{-1}$ (finite!), then
\[ \int hdm \leq K \sup x \cdot \|m\|_T, \]
so that $M_T$ is topologically contained in $C_0'$. Now if $A_S \subseteq A_T$, then $M_S \subseteq M_T$, and the latter (hence the former) inclusion is topological, since both spaces are Banach spaces topologically contained in $C_0'$ (cf. the Closed Graph Theorem). Since $f_i \in A_S$, it follows that
\[ \|T(t)\| = \|f_i\|_T \leq B\|f_i\|_S = B\|S(t)\|, \quad t \in R. \]
This proves the nontrivial part of (i), and (ii) is proved by the same argument.

2. The group $T(\cdot)$ is bounded away from zero if $\inf \|T(\cdot)\| > 0$, and temperate if $\|T(t)\| = O(|t|^k)$ as $|t| \to \infty$, for some nonnegative integer $k$. Let
\[ w^\pm = \lim_{t \to \pm \infty} t^{-1} \log \|T(t)\| \]
(cf. [2, p. 244]). The spectral norm of $T(t)$ is $\exp[tw^+]$ for $t \geq 0$ and $\exp[tw^-]$ for $t < 0$; this implies the equivalence of the following statements:
1. $T(\cdot)$ is bounded away from zero;
2. $\|T(\cdot)\| \geq 1$;
3. $w^- \leq 0 \leq w^+$. 

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In particular, *temperate groups are bounded away from 0.*

For $Q \subset R$ compact, denote by $A_T(Q)$ the closed subspace of $A_T$ consisting of all $f \in A_T$ with support in $Q$. If $f = \hat{m} \in A_T(Q)$, then
\[ dm(t) = \hat{f}(t) dt/2\pi, \]
and therefore
\[ ||f||_T = \int ||T(t)|| ||\hat{f}(-t)|| dt/2\pi, \quad f \in A_T(Q). \]

**Theorem 2.** Let $S(\cdot)$ and $T(\cdot)$ be strongly continuous groups of operators. If $S(\cdot)$ is bounded away from 0 and $||T(t)||/||S(t)|| \to 0$ as $|t| \to \infty$, then, for each compact $Q \subset R$ and $x \in X$, $A_S(Q)$ is compactly imbedded in $A_T$ and the restrictions $\tau \cdot A_S(Q)$ and $\tau(\cdot)x A_S(Q)$ are compact operators. Conversely, if $S(\cdot)$ and $T(\cdot)$ are temperate and $A_S(Q)$ is compactly imbedded in $A_T$ for some compact $Q$ with nonvoid interior, then $||T(t)||/||S(t)|| \to 0$ as $|t| \to \infty$.

**Proof.** 1. Let $S(\cdot)$ be bounded away from 0 (hence $||S(\cdot)|| \geq 1$), and let $||T(t)||/||S(t)|| \to 0$ as $|t| \to \infty$. Given $\varepsilon > 0$, choose $N > 0$ such that $||T(t)||/||S(t)|| < \varepsilon$ for $|t| > N$. Let $Q \subset R$ be compact, and let $\{h_n\}$ be in the closed unit ball of $A_S(Q)$. By (3), $\{h_n\}$ is in the closed $2\pi$-ball of $L^1(R)$. Therefore, a subsequence $\{h'_n\}$ converges weak* in $(L^\infty)^*$. In particular, for each $s \in R$
\[ \frac{1}{2\pi} \int e^{-ist} h'_n(t) dt = h_n(s) \]
converge. Since
\[ |h_n(s)| \leq \frac{1}{2\pi} ||h'_n|| \leq 1 \quad \text{and} \quad h_n(t) = \int_Q e^{ist} h'_n(s) ds, \]
it follows by Dominated Convergence that $h'_n$ converge pointwise on $R$ and $|h_n| \leq |Q|$ (the Lebesgue measure of $Q$). Now, by (3),
\[ ||h_n - h_m||_T \leq \int_{-N}^{N} ||T(t)|| \left| h'_n(t) - h'_m(t) \right| dt/2\pi + \varepsilon ||h_n - h_m||_T. \]
The first term on the right tends to 0 as $n, m \to \infty$ (Dominated Convergence), and the second term is $\leq 2\varepsilon$. Hence $\{h_n\}$ is Cauchy in $A_T$, and the first part of the theorem follows.

2. Let $S(\cdot)$ and $T(\cdot)$ be temperate, and suppose $A_S(Q)$ is compactly imbedded in $A_T$ for some compact $Q$ with nonvoid interior. There exists a nonzero $C^\infty$ function $g$ with support in $Q$. Let $g_t = g(t)/||S(t)|| (t \in R)$. Then $g_t \in A_S(Q)$, and by (3),

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\[ \|g_t\|_S = \int \|S(s)\| |g(t - s)| \|S(t)\|^{-1} ds / 2\pi \]
\[ = \int \|S(t + u)\| \|S(t)\|^{-1} |g(-u)| du / 2\pi \]
\[ \leq \int \|S(u)\| |g(-u)| du / 2\pi = \|g\|_S \quad (t \in R). \]

Therefore, the set \( \{g_t| t \in R\} \) has limit points in \( A_T \), which are also limit points in the topology induced on \( A_T \) by the weak* topology on \( L^\infty(R) \) (because \( \|T(\cdot)\| \geq 1 \)). However, \( g_t \to 0 \) (as \( |t| \to \infty \)) in the latter topology, since for each \( h \in L^1(R) \),

\[ \left| \int h(s)g_s(s) ds \right| = \left| \langle hg, \hat{\chi} \rangle (t) \right| / \|S(t)\| \leq \|\langle hg, \hat{\chi} \rangle (t)\| \to 0 \]

by the Riemann-Lebesgue lemma. Therefore \( g_t \to 0 \) in \( A_T \) as \( |t| \to \infty \).

Now

\[ \|g_t\|_T = \int \|T(s)\| |g(t - s)| ds / (2\pi \|S(t)\|) \]
\[ \geq (\|T(t)\| / \|S(t)\|) \int \|T(t - s)\|^{-1} |g(t - s)| ds / 2\pi \]
\[ = c \|T(t)\| / \|S(t)\|, \]

where \( c = \int \|T(u)\|^{-1} |g(u)| du / 2\pi \) is a positive constant. Since \( \|g_t\|_T \to 0 \), the result follows.

The same arguments (with the estimate \( \|T(s)x\| \geq \|T(t)x\| \cdot \|T(t - s)\|^{-1} \) replacing the inequality \( \|T(s)x\| \geq \|T(t)x\| \cdot \|T(t - s)\|^{-1} \) in the last part of the proof) prove the following.

**Theorem 3.** Let \( T(\cdot) \) be a strongly continuous group of operators, and let \( x \in X \). If \( T(\cdot) \) is bounded away from zero and \( x \) is nonregular, then \( A_T(Q) \) is compactly imbedded in \( A_{Tz} \) for each compact \( Q \subset R \). Conversely, if \( T(\cdot) \) is temperate and \( A_T(Q) \) is compactly imbedded in \( A_{Tz} \) for some compact set \( Q \) with nonvoid interior, then \( x \) is nonregular.

**Corollary.** Let \( T(\cdot) \) be a temperate strongly continuous group of operators. Then \( x \) is a regular vector for \( T(\cdot) \) if and only if the injection \( A_T(Q) \to A_{Tz} \) is not compact for any compact set \( Q \) with nonvoid interior.

**3. Remarks.** The existence of regular vectors seems to be a difficult question in general. Observe that if \( S(\cdot) \) and \( N(\cdot) \) are commuting
groups of operators, and \( S(\cdot) \) is uniformly bounded, then the groups 
\( T(\cdot) = S(\cdot)N(\cdot) \) and \( N(\cdot) \) have the same (strongly) regular vectors. 
This follows from the estimate (with \( K = \sup_\xi \| S(\cdot) \| ):
\[
\left\| T(t)x \right\|/\| T(t) \| \geq K^{-1} \left\| S(t)N(t)x \right\|/\| N(t) \| \geq K^{-2} \left\| N(t)x \right\|/\| N(t) \|.
\]
In particular, every nonzero vector is strongly regular for a uniformly bounded group.

**Proposition.** If \( T \) is a real spectral operator of finite type, then the group \( T(t) = e^{itT} \) possesses regular vectors.

**Proof.** Let \( T = S+N \) be the canonical decomposition of \( T \) (cf. [1]). Then \( T(\cdot) = S(\cdot)N(\cdot) \), where \( S(t) = e^{itS} \) and \( N(t) = e^{itN} \) are commuting groups and \( S(\cdot) \) is uniformly bounded. By the preceding remark, it suffices to show that \( N(\cdot) \) possesses regular vectors. Let \( k \) be the nonnegative integer defined by the requirements \( N^k \neq 0 \) and \( N^{k+1} = 0 \). Then
\[
(it)^{-k}N(t) = \sum_{j=0}^{k-1} (it)^{-j}N^j/j! + N^k/k! \rightarrow N^k/k! 
\]
as \( |t| \rightarrow \infty \), in the uniform operator topology. Therefore,
\[
\lim_{|t| \rightarrow \infty} \left\| N(t)x \right\|/\| N(t) \| = \lim_{|t| \rightarrow \infty} \left\| (it)^{-k}N(t)x \right\|/\| (it)^{-k}N(t) \|
\]
\[
= \| N^kx \|/\| N^k \|,
\]
so that each vector not in the null space of \( N^k \) is regular for \( N(\cdot) \).

**Conjecture.** Every temperate group of operators possesses regular vectors.

The following example of R. Beals shows that the “finite type” hypothesis cannot be omitted in the above proposition.

For \( n = 1, 2, \cdots \), let \( N_n \) be the \( n \times n \) matrix with \( 1/n \) in the first superdiagonal and 0 elsewhere, and consider \( N = \sum \oplus N_n \) as an operator in \( \ell_1 \). Since \( N_n^* = 0 \) and \( \| N_n^k \| = n^{-k} \) for \( k < n \), we have \( N^k = \sum_{n > k} \oplus N_n^k \) and \( \| N^k \| = \sup_{n > k} n^{-k} = (k + 1)^{-k} \). Hence \( N \) is quasinilpotent. Let \( N(t) = e^{tN} = \sum \oplus \exp(tN_n) \). One calculates that
\[
\| \exp(tN_n) \| = \sup_n \| \exp(tN_n) \|/\| \exp(tN_n + t) \| \rightarrow 0\] as \( t \rightarrow \infty \).
Let $P_n(x_1, x_2, \ldots) \rightarrow (x_1, x_2, \ldots, x_{m(n)}, 0, 0, \ldots)$, where $m(n) = 1 + \ldots + n$. For any $x \in l_2$ and $n = 1, 2, \ldots$,
\[
\|N(t)x\|/\|N(t)\| \leq \|N(t)P_n x\|/\|N(t)\| + \|N(t)(I - P_n)x\|/\|N(t)\|
\]
\[
\leq \exp(tN_n)\|x\|/\|N(t)\| + \|(I - P_n)x\|,
\]
and therefore,
\[
\limsup_{t \rightarrow \infty} \|N(t)x\|/\|N(t)\| \leq \|(I - P_n)x\|,
\]
by (1). Letting $n \rightarrow \infty$, we obtain that the lim sup is 0. Similarly, the lim sup as $t \rightarrow -\infty$ is 0. Since $x$ was arbitrary, this shows that $N(\cdot)$ has no regular vectors.

References


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