COMPLETENESS OF HAMILTONIAN VECTOR FIELDS

DAVID G. EBIN

ABSTRACT. We prove that under certain conditions the flow of a Hamiltonian vector field on a possibly infinite-dimensional dynamical system exists for all time.

We shall prove that under certain natural conditions the Hamiltonian vector field of a (possibly infinite-dimensional) dynamical system has integral curves which extend for all time. W. Gordon [3] has proved the result in the case of Euclidean space. We are grateful to J. Marsden for suggesting the problem and to W. Meyer for useful conversations.

Let M be a smooth Hilbert manifold with a Riemannian structure, $\langle\ ,\ \rangle$, let $V_0: M \to R$ be a smooth function and let $V = V_0 \circ \pi$ where $\pi: T(M) \to M$ is the projection map of the tangent bundle. Define $H: T(M) \to R$ by $H = K + V$ where K, the kinetic energy, is defined by $K(V) = \frac{1}{2}\langle V,\ V\rangle$.

It is well known that T^*M (and by means of $\langle\ ,\ \rangle$, TM) has a natural symplectic two-form Ω, and the equation $i(Z)\Omega = dH$ can be used to define a vector field Z on TM [4, p. 110].

THEOREM. *If M is complete under the metric induced by $\langle\ ,\ \rangle$ and if V is bounded below, then all integral curves of Z extend for all time. Equivalently the flow $F: TM \times R \to TM$ which satisfies $(\partial F/\partial t)(X,\ s) = Z_{F(X,s)}$ is defined on all of $TM \times R$.*

To prove the theorem we must construct a Riemannian structure for TM and this we do by use of the affine connection.

It is well known that $\langle\ ,\ \rangle$ defines a unique connection and for each $p \in M$ and $X \in T_pM$, ∇ defines a splitting of T_XTM into vertical and horizontal subspaces; i.e. $T_XTM = \mathcal{V} \oplus \mathcal{H}$, [2]. Also there are canonical linear isomorphisms $L: \mathcal{V} \to T_pM$ and $T\pi: \mathcal{H} \to T_pM$, $T\pi$ being the tangent map to π. We define an inner product, $\langle\langle\ ,\ \rangle\rangle$, on T_XTM by declaring $\mathcal{V}$ and $\mathcal{H}$ perpendicular and L and $T\pi$ isometries. This clearly defines a Riemannian structure for $T(M)$.

LEMMA. *$\langle\langle\ ,\ \rangle\rangle$ induces a complete metric on TM.*

Received by the editors February 23, 1970.

AMS 1969 *subject classifications*. Primary 3465, 3480; Secondary 7040.

Key words and phrases. Hamiltonian vector field, Hilbert manifold, Riemannian structure, symplectic two-form, vertical subspace, horizontal subspace, spray, complete metric.

PROOF. Let $\{X_i\}$ be a Cauchy sequence in TM and let $p_i=\pi(Xi)$. For any $q\in M$ and $y\in T_qM$, $T\pi: T_yTM\to T_qM$ is distance decreasing. Hence π decreases the length of curves, so $\{p_i\}$ is Cauchy in M.

Let $p=\lim_{i\to\infty}\{p_i\}$ and choose a closed neighborhood U about p and a chart $\varphi: U\to E$. (E is a Hilbert space on which M is modeled.) φ induces a chart $T\varphi: T(U)\to E\times E$ in a standard way [4, p. 40], and for i sufficiently large $X_i\in T(U)$. We can assume that $\varphi(U)$ is a closed ball in E in which case $T\varphi(T(U))=\varphi(U)\times E$ is a closed subset of $E\times E$. $\varphi(U)\times E$ is therefore complete as a metric subspace of $E\times E$, so if we can show $\{T\varphi(X_i)\}$ is Cauchy in $\varphi(U)\times E$ then $\{T\varphi(X_i)\}$ will converge and so will $\{X_i\}$.

To show $\{T\varphi(X_i)\}$ Cauchy, we make $\varphi: U\to E$ a normal coordinate chart so that $TT\varphi: T_yTM\to U\times E\times E\times E$ is an isometry for all $y\in\pi^{-1}(p)$, and so that there exists a constant A such that for any $W\in TTU$, $\|TT\varphi W\|\leqq A\|W\|$. The existence of such an A means that all curves in TU are stretched, under the map $T\varphi$, by at most a factor of A. Then, letting ρ be the metric on TM, we must show that for i, j sufficiently large and for any curve γ of length sufficiently close to $\rho(X_i, X_j)$, γ is contained in TU. If this is true, $T\varphi(\gamma)$ has length at most A times the length of γ and the distance between $T\varphi(X_i)$ and $T\varphi(X_j)$ is at most $A\rho(X_i, X_j)$.

To show curves such as γ are contained in $T(U)$ we assume U is a metric ball about p of radius 3ϵ. Then let $V\subseteq U$ be the ball of radius ϵ, and pick i, j sufficiently large so that $\rho(X_i, X_j)<\epsilon$ and $X_i, X_j\in T(V)$. Then if γ is a curve from X_i to X_j of length less than 2ϵ, $\gamma_0=\pi\circ\gamma$ must also have length less than 2ϵ so $\gamma_0\subseteq U$. Thus $\gamma\leqq TU$, $T\varphi(\gamma)$ has length at most $2A\epsilon$, and the distance from $T\varphi(X_i)$ to $T\varphi(X_j)$ is therefore at most 2ϵ. Hence $\{T\varphi(X_i)\}$ is Cauchy and the lemma is proven.

PROOF OF THEOREM. Consider the equation $i(Z)\Omega=dH$ and let $Z=S+G$ where S satisfies $i(S)\Omega=dK$ and G satisfies $i(G)\Omega=dV$. Then S is the spray of the Riemannian metric $\langle\ ,\ \rangle$ [4, p. 110], and G is the vertical lift of grad V; i.e. $G_x=L^{-1}(\text{grad } V)$ where $L:\mathcal{V}\to T_{\pi(x)}M$. (See [1].)

Let $c:(a, b)\to TM$ be a maximal integral curve of Z. We wish to show that $a=-\infty$, $b=\infty$, and to do this, we first assume a (respectively b) finite and show $\lim_{t\to a}\{c(t)\}$ (respectively $\lim_{t\to b} c(t)$) exists in TM. Then by the fundamental theorem of ordinary differential equations [4], the domain of c extends to $(a+\epsilon, b)$ (respectively $(a, b+\epsilon)$). This contradicts the maximality of (a, b) and proves the theorem.

The main fact needed to show $\lim_{t\to a}\{c(t)\}$ exists, is that H is constant along c. This is simply conservation of energy and is proven

in [1, p. 110]. Let N be the lower bound for V and let $N_1 = H(c)$. Then $K \restriction c$ is bounded by $N_1 - N$.

Let $c_0 = \pi \circ c$, a curve in M. Then c_0', the tangent vector to c_0 satisfies $c_0'(t) = T\pi(c'(t)) = T\pi(Z_{c(t)})$. But $T\pi(Z_{c(t)}) = T\pi(S_{c(t)}) = c(t)$ since G is vertical and S is a spray. Therefore since K is bounded on c, the length of $c_0'(t)$ is bounded and, since M is complete, $\lim_{t\to b} \{c_0(t)\}$ exists. Let $c_0(b) = \lim_{t\to b} \{c_0(t)\}$.

Now we seek a limit of $c(t)$ in $\pi^{-1}(c_0(b))$. To show that it exists we must only show that $c'(t)$ is bounded in the metric on TM. $c'(t) = Z_{c(t)} = S_{c(t)} + G_{c(t)}$, and S and G are horizontal and vertical respectively. Also $G_{c(t)}$ depends only on $c_0(t)$ and since $c_0(t)$ extends continuously to $c_0(b)$, $G_{c(t)}$ remains bounded as $t \to b$. Also $\|S_{c(t)}\| = \|c(t)\|$, from the definition of the metric on TM. It follows that $\|c'(t)\| = \|S_{c(t)}\| + \|G_{c(t)}\|$ remains bounded as $t \to b$, so $c(t)$ converges. The same argument works as $t \to a$, and the theorem follows.

REMARK 1. If M is finite-dimensional, one can use local compactness of TM instead of a metric on TM, so the proof is much shorter.

REMARK 2. It is easy to show that the theorem is false if M is not complete or V is not bounded below. There are counterexamples even if M is one-dimensional.

REFERENCES

1. R. Abraham, *Foundations of mechanics*, Benjamin, New York, 1967. MR **36** #3527.

2. P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88. MR **25** #4463.

3. W. Gordon, *On completeness of vector fields* (preprint).

4. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962. MR **27** #5192.

STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11970.