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Abstract. Let $aR$ denote the principal right ideal generated in a ring $R$ by an element $a$. Kaplansky has raised the question: If $aR = bR$, are $a$ and $b$ necessarily right associates? In this note we show that for rings of continuous functions the answer is affirmative if and only if the underlying topological space is zero-dimensional. This gives an algebraic characterization of the topological concept "zero-dimensional". By extending the notion of uniqueness of generators of principal ideals we are able to give an algebraic characterization of the concept "$n$-dimensional".

In the sequel, $R$ is assumed to be commutative.

Definition. A set of principal ideals \( \{a_iR\}, \ i = 1, \ldots, n, \) is uniquely generated if whenever $a_iR = b_iR$, $i = 1, \ldots, n$, there exist elements $u_i$ of $R$ such that $a_i = b_iu_i$, $i = 1, \ldots, n$, and $u_1R + \cdots + u_nR = R$. The dimension of $R$—denoted by $\dim R$—is the least integer $n$ such that every set of $n+1$ principal ideals is uniquely generated.

In the following $X$ will denote a completely regular topological space, $C(X)$ the ring of real-valued continuous functions defined on $X$, and $C^*(X)$ the subring of $C(X)$ consisting of the bounded functions in $C(X)$. For $f \in C(X)$, the zero set $Z(f)$ of $f$, is defined by $Z(f) = \{x \in X : f(x) = 0\}$. Clearly $f_1C(X) + \cdots + f_nC(X) = C(X)$ if and only if $Z(f_1) \cap \cdots \cap Z(f_n) = \emptyset$. For further information on zero sets the reader is referred to [2]. An important fact is that disjoint zero sets are completely separated.

We use the modification of covering dimension involving basic covers [2, p. 243], and the equivalent definitions given in [1]. The unit cube in $E^{n+1}$ is denoted by

\[ I^{n+1} = \{x \in E^{n+1} : -1 \leq x_i \leq 1, i = 1, \ldots, n+1\}. \]

We also write

\[ I_+^{n+1} = \{x \in E^{n+1} : 0 \leq x_i \leq 1, i = 1, \ldots, n+1\} \]

and

\[ S_n^+ = \{x \in I_+^{n+1} : x_i = 0 \text{ or } 1 \text{ for some } i\}. \]

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Theorem. The following conditions are equivalent.

(i) \( \text{dim } X = n \).

(ii) \( \text{dim } C(X) = n \).

(iii) \( \text{dim } C^*(X) = n \).

Proof. \( \text{dim } C(X) \leq \text{dim } X \). Let \( \text{dim } X = n \) and suppose \( f_i C(X) = g_i C(X) \), \( i = 1, \ldots, n+1 \). There exist functions \( s_i, t_i \) in \( C(X) \) such that \( f_i = g_i s_i, g_i = f_i t_i \), \( i = 1, \ldots, n+1 \). Then \( g_i = g_i s_i t_i \) so that \( g_i(1 - s_i t_i) = 0 \).

Since \( Z_i = Z(s_i) \) and \( Z'_i = Z(s_i + 1 - s_i t_i) \) are disjoint zero sets, there exist \( m_i \) in \( C(X) \) such that \( m_i \) is 1 on \( Z_i \), 0 on \( Z'_i \) and \( 0 \leq m_i \leq 1 \).

Let \( m \) be the mapping of \( X \) into \( \mathbb{R}^{n+1} \) defined by \( m(x) = (m_1(x), \ldots, m_{n+1}(x)) \). Then \( m \) maps all the points in \( Z_i \) and \( Z'_i \) into \( \mathbb{S}^n \). Since \( \text{dim } X \leq n \), there exists a mapping \( h = (h_1, \ldots, h_{n+1}) \) of \( X \) into \( \mathbb{S}^n \) such that \( h(x) = m(x) \) whenever \( m(x) \in \mathbb{S}^n \) [1, Definition 3]. Let \( k_i = s_i + h_i(1 - s_i t_i), i = 1, \ldots, n+1 \). Then \( Z(k_i) \cap \cdots \cap Z(k_{n+1}) = \emptyset \).

To see this, note that for any \( x \in X \) we have \( h_i(x) = 0 \) or \( h_i(x) = 1 \) for some \( i \). If \( h_i(x) = 0 \) then \( s_i(x) \neq 0 \) so that \( k_i(x) \neq 0 \). If \( h_i(x) = 1 \), then \( s_i(x) + 1 - s_i(x) t_i(x) \neq 0 \) and again \( k_i(x) \neq 0 \). Hence \( k_i C(X) + \cdots + k_{n+1} C(X) = C(X) \). Finally, \( g_i k_i = g_is_i + h_i g_i(1 - s_i t_i) = g_is_i = f_i, i = 1, \ldots, n+1 \).

\( \text{dim } X \leq \text{dim } C(X) \). Let \( \text{dim } C(X) = n \) and let \( Z_i, Z'_i, i = 1, \ldots, n+1 \), be disjoint pairs of zero sets of \( X \). We construct functions \( f_i, k_i \) such that \( f_i = k_i f_i \) and \( f_i = 1 \) on \( Z_i, f_i = -1 \) on \( Z'_i, i = 1, \ldots, n+1 \). There exist \( p_i \) such that \( p_i = 1 \) on \( Z_i \) and \( p_i = -1 \) on \( Z'_i \). There exist \( k_i, s_i, t_i \) such that \( k_i = 1 \) when \( p_i \geq \frac{1}{2}, k_i = -1 \) when \( p_i \leq -\frac{1}{2}, -1 \leq k_i \leq 1 \):

\[
\begin{align*}
t_i &= 1 \quad \text{on } Z_i, t_i = 0 \quad \text{when } p_i \leq \frac{1}{2}, 0 \leq t_i \leq 1, \\
s_i &= -1 \quad \text{on } Z'_i, s_i = 0 \quad \text{when } p_i \geq -\frac{1}{2}, -1 \leq s_i \leq 0.
\end{align*}
\]

Let \( f_i = s_i + t_i \). Then \( f_i = k_i f_i \), \( f_i = k_i f_i \), so that \( f_i C(X) = |f_i| C(X), i = 1, \ldots, n+1 \). Since \( \text{dim } C(X) \leq n \), there exist \( h_1, \ldots, h_{n+1} \) such that \( f_i = h_i |f_i|, i = 1, \ldots, n+1 \), and \( h_1 C(X) + \cdots + h_{n+1} C(X) = C(X) \). Clearly \( Z_i \) and \( Z'_i \) are separated in \( X - Z(h_i), Z(h_i) \cap \cdots \cap Z(h_{n+1}) = \emptyset \). Hence \( \text{dim } X \leq n \) [1, Definition 4].

\( \text{dim } X = \text{dim } C^*(X) \). This is proved using the same methods as above, with \( C(X) \) replaced by \( C^*(X) \).

A simple consequence of this theorem is the following well-known result.

Corollary. \( \text{dim } X = \text{dim } \beta X = \text{dim } vX \), where \( \beta X \) is the Stone-Čech compactification of \( X \), and \( vX \) is the Hewitt real-compactification of \( X \).

Proof. Since \( C^*(X) \) is isomorphic to \( C(\beta X) \), we have \( \text{dim } X \)
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\[ n \iff \dim C^*(X) = n \iff \dim C(\beta X) = n \iff \dim \beta X = n. \]
Since \( C(X) \) is isomorphic to \( C(\nu X) \), we have \( \dim X = n \iff \dim C(X) = n \iff \dim C(\nu X) = n \iff \dim \nu X = n. \)

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