UNIQUENESS OF GENERATORS OF PRINCIPAL IDEALS
IN RINGS OF CONTINUOUS FUNCTIONS

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Abstract. Let \( aR \) denote the principal right ideal generated in a ring \( R \) by an element \( a \). Kaplansky has raised the question: If \( aR = bR \), are \( a \) and \( b \) necessarily right associates? In this note we show that for rings of continuous functions the answer is affirmative if and only if the underlying topological space is zero-dimensional. This gives an algebraic characterization of the topological concept "zero-dimensional". By extending the notion of uniqueness of generators of principal ideals we are able to give an algebraic characterization of the concept "\( n \)-dimensional".

In the sequel, \( R \) is assumed to be commutative.

Definition. A set of principal ideals \( \{a_iR\}, \ i = 1, \ldots, n \), is uniquely generated if whenever \( a_iR = b_iR, \ i = 1, \ldots, n \), there exist elements \( u_i \) of \( R \) such that \( a_i = b_iu_i, \ i = 1, \ldots, n \), and \( u_1R + \cdots + u_nR = R \). The dimension of \( R \)—denoted by \( \text{dim} \ R \)—is the least integer \( n \) such that every set of \( n+1 \) principal ideals is uniquely generated.

In the following \( X \) will denote a completely regular topological space, \( C(X) \) the ring of real-valued continuous functions defined on \( X \), and \( C^*(X) \) the subring of \( C(X) \) consisting of the bounded functions in \( C(X) \). For \( f \in C(X) \), the zero set \( Z(f) \) of \( f \), is defined by \( Z(f) = \{ x \in X : f(x) = 0 \} \). Clearly \( f_1C(X) + \cdots + f_nC(X) = C(X) \) if and only if \( Z(f_1) \cap \cdots \cap Z(f_n) = \emptyset \). For further information on zero sets the reader is referred to [2]. An important fact is that disjoint zero sets are completely separated.

We use the modification of covering dimension involving basic covers [2, p. 243], and the equivalent definitions given in [1]. The unit cube in \( E^{n+1} \) is denoted by

\[
I^{n+1} \cap I^{n+1} = \{ x \in E^{n+1} : \ -1 \leq x_i \leq 1, \ i = 1, \ldots, n + 1 \}.
\]

We also write

\[
I_+^{n+1} = \{ x \in E^{n+1} : 0 \leq x_i \leq 1, \ i = 1, \ldots, n + 1 \}
\]

and

\[
S^n_+ = \{ x \in I_+^{n+1} : x_i = 0 \text{ or } 1 \text{ for some } i \}.
\]
A simple consequence of this theorem is the following well-known result.

**Corollary.** \( \dim X = \dim \beta X = \dim vX \), where \( \beta X \) is the Stone-Čech compactification of \( X \), and \( vX \) is the Hewitt real-compactification of \( X \).

**Proof.** Since \( C^*(X) \) is isomorphic to \( C(\beta X) \), we have \( \dim X \)
\[ n \iff \dim C^*(X) = n \iff \dim \beta X = n. \] Since \( C(X) \) is isomorphic to \( C(\nu X) \), we have \( \dim X = n \iff \dim C(X) = n \iff \dim C(\nu X) = n \iff \dim \nu X = n. \]

**References**


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