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A DETERMINANTAL INEQUALITY FOR PROJECTORS IN A UNITARY SPACE[1]

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ABSTRACT. Let V be a finite dimensional unitary space and $V=V_1+\cdots+V_k$ a direct decomposition. Let P_i be the orthogonal projector in V with range V_i. If $A=P_1+\cdots+P_k$ we prove that $0<\det(A)\leqq 1$ and $\det(A)=1$ if and only if $V=V_1+\cdots+V_k$ is an orthogonal decomposition.

Let N_i $(1\leqq i\leqq k)$ be a normal operator in V, of rank r_i. Assume that $N=\sum_{i=1}^k N_i$ has rank $r=r_1+\cdots+r_k\leqq n=\dim V$. If the nonzero eigenvalues of N (counting multiplicities) are the same as the nonzero eigenvalues of all N_i $(1\leqq i\leqq k)$ together, then $N_iN_j=0$ for $i\neq j$. This generalizes a recent result of L. Brand.

I. **Introduction.** The following theorem was proved recently by L. Brand [1]:

THEOREM 1. *Let A and B be real $n\times n$ symmetric matrices with $\lambda_1,\cdots,\lambda_r,0,\cdots,0$ and $0,\cdots,0,\lambda_{r+1},\cdots,\lambda_n$ $(\lambda_i\neq 0,\ 1\leqq i\leqq n)$ as eigenvalues, respectively. If $A+B$ has eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ then $AB=0$.*

One of the aims of this paper is to generalize this theorem. We shall prove that the assertion remains valid when A and B are complex normal $n\times n$ matrices. Moreover, we shall prove that an analogous result holds for sums of several normal matrices (Theorem 3).

In this paper V denotes n-dimensional complex unitary space with scalar product (x, y). Our Theorem 2 gives a necessary and sufficient condition for a direct sum $V=V_1+\cdots+V_k$ to be an orthogonal direct sum.

II. **Sum of projectors.** We begin by proving three lemmas:

LEMMA 1. *Let A be an operator in V such that:*
(i) *all eigenvalues of A have modulus one,*
(ii) $\|Ax\|\leqq\|x\|$ *for all $x\in V$.*
Then A is a unitary operator.

PROOF. In view of (i), it is sufficient to show that A is normal.

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We claim that the eigenvectors of A span V. If this is false, there exists an eigenvector x of A such that $Ax=\lambda x$, $\|x\|=1$, $Ay=\lambda y+x$ for some $y\in V$. For $z=y-(y, x)x$ we have $Az=\lambda z+x$, $\|Az\|^2=\|z\|^2+1$ which contradicts (ii). This proves that our claim is true.

We shall prove that A is normal by showing that $Ax=\lambda x$, $Ay=\mu y$, $\lambda\neq\mu$ imply that $(x, y)=0$. Assume that $(x, y)\neq 0$. Replacing x by θx where θ is a suitable nonzero complex number, we can further assume that the real part of $(\lambda\bar{\mu}-1)(x, y)$ is positive. Then

$$\begin{aligned}\|A(x+y)\|^2-\|x+y\|^2 &= \|\lambda x+\mu y\|^2-\|x+y\|^2\\ &= 2\operatorname{Re}\{(\lambda\bar{\mu}-1)(x, y)\} > 0\end{aligned}$$

contradicts (ii). Therefore we must have $(x, y)=0$.

The lemma is proved.

LEMMA 2. *Let $V=V_1+V_2$ be a direct sum, $V_i\neq 0$, $i=1, 2$. Let P_i $(i=1, 2)$ be the orthogonal projector in V with range V_i and*

$$P_{12}: V_1\to V_2 \quad \text{and} \quad P_{21}: V_2\to V_1$$

be the restrictions of P_2 and P_1, respectively. Then

$$\det(I_2-P_{12}P_{21})\leqq 1 \tag{1}$$

where I_2: $V_2\to V_2$ is the identity operator. The equality holds in (1) *if and only if V_1 is orthogonal to V_2.*

PROOF. Since P_1 and P_2 are hermitian positive semidefinite the same is true for $B=P_1+P_2$. If $x\in V$ and $Bx=P_1x+P_2x=0$ then $P_1x=P_2x=0$ since $V=V_1+V_2$ is a direct sum. We infer that x is orthogonal to V_1 and V_2 and consequently $x=0$. Hence, B is in fact positive definite. If we choose bases in V_1 and V_2 they form a base of V. In such a base the matrix of B is

$$B=\left(\begin{array}{c|c} I_1 & P_{21}\\ \hline P_{12} & I_2\end{array}\right)$$

where I_1, I_2, P_{12}, P_{21} are matrices of I_1, I_2, P_{12}, P_{21}, respectively (I_1: $V_1\to V_1$ is the identity operator). We have

$$\det(A)=\det\left(\begin{array}{c|c} I_1 & 0\\ \hline P_{12} & I_2-P_{12}P_{21}\end{array}\right)=\det\left(\begin{array}{c|c} I_1 & P_{21}\\ \hline P_{12} & I_2\end{array}\right)=\det(B)>0.$$

If $x\in V_2$ then

$$\begin{aligned}\|Ax\|^2=(Ax, Ax) &= (x-P_2P_1x, x-P_2P_1x)\\ &= \|x\|^2-(P_2P_1x, x)-(x, P_2P_1x)+\|P_2P_1x\|^2\\ &= \|x\|^2-2\|P_1x\|^2+\|P_2P_1x\|^2\end{aligned}$$

because $(P_2P_1x,\ x)=(P_1x,\ P_2x)=(P_1x,\ x)=\|P_1x\|^2$ and $(x,\ P_2P_1x)=\|P_1x\|^2$. It follows that

$$\|Ax\|^2 \leqq \|x\|^2 - \|P_1x\|^2 \tag{2}$$

since $\|P_1x\|^2-\|P_2P_1x\|^2\geqq 0$. We infer that $|\lambda|\leqq 1$ for every eigenvalue λ of A. Hence, $|\det(A)|\leqq 1$ which together with $\det(A)>0$ establishes (1).

If $\det(A)=1$ then every eigenvalue of A must have modulus 1. By Lemma 1 and (2), A is unitary. Now, (2) implies that $P_1x=0$ for all $x\in V_2$, i.e., V_1 and V_2 are orthogonal to each other. The converse is trivial.

LEMMA 3. *Let $V=V_1+V_2$ be a direct sum, $V_i\neq 0$, $i=1, 2$. Let H be a hermitian operator in V with range V_1, such that its restriction H_1 to V_1 is positive definite, and let P_2 be the orthogonal projector in V with range V_2. Then*

$$0 < \det(H + P_2) \leqq \det(H_1). \tag{3}$$

The equality holds in (3) *if and only if V_1 is orthogonal to V_2.*

PROOF. We fix bases in V_1 and V_2. They form together a basis of V. All matrices will be taken with respect to these bases. The matrix of $H+P_2$ has the form

$$\left(\begin{array}{c|c} H_1 & H_1P_{21} \\ \hline P_{12} & I_2 \end{array}\right)$$

where:

H_1 is the matrix of H_1,
P_{12} is the matrix of the orthogonal projector $P_{12}: V_1\to V_2$,
P_{21} is the matrix of the orthogonal projector $P_{21}: V_2\to V_1$,
I_2 is the identity matrix.

Since

$$\det\left(\begin{array}{c|c} H_1 & H_1P_{21} \\ \hline P_{12} & I_2 \end{array}\right) = \det\left(\begin{array}{c|c} H_1 & 0 \\ \hline P_{12} & I_2 - P_{12}P_{21} \end{array}\right)$$

we get

$$\det(H + P_2) = \det(H_1)\cdot\det(I_2 - P_{12}P_{21}).$$

It remains to apply Lemma 2.

THEOREM 2. *Let*

$$V = V_1 + \cdots + V_k \tag{4}$$

be a direct sum, $V_i \neq 0$. *Let* P_i *be the orthogonal projector in* V *with range* V_i *and let* $A = P_1 + \cdots + P_k$. *Then* $0 < \det(A) \leqq 1$ *and* $\det(A) = 1$ *if and only if* (4) *is an orthogonal sum.*

PROOF. A is hermitian positive definite (the argument is the same as for B in the proof of Lemma 2). Hence, $\det(A) > 0$.

We shall prove the remaining assertions of the theorem by induction on k. If $k=1$, they are trivially true. For $k>1$ we put

$$U = V_1 + \cdots + V_{k-1}. \tag{5}$$

Let Q_i $(1 \leqq i \leqq k-1)$ be the restrictions of P_i to U and $H_1 = Q_1 + \cdots + Q_{k-1}$. By inductive hypothesis and the first part of the proof, we conclude that $0 < \det(H_1) \leqq 1$ and $\det(H_1) = 1$ if and only if (5) is an orthogonal sum. We can apply Lemma 3 with $H = P_1 + \cdots + P_{k-1}$. It follows that $0 < \det(A) = \det(H + P_k) \leqq \det(H_1) \leqq 1$.

If $\det(A) = 1$ then V_k and U are orthogonal to each other by Lemma 3. Since (5) is also an orthogonal sum, the theorem is proved.

III. **Sum of normal operators.** Now we are able to prove the following:

THEOREM 3. *Let* N_i, $1 \leqq i \leqq k$, *be normal operators in* V *with eigenvalues*

$$\lambda_1^{(i)}, \cdots, \lambda_{r_i}^{(i)}, 0, \cdots, 0,$$

where $\lambda_j^{(i)} \neq 0$, $1 \leqq j \leqq r_i$, $1 \leqq i \leqq k$ *and* $r_1 + \cdots + r_k \leqq n$. *If the operator* $N = N_1 + \cdots + N_k$ *has eigenvalues*

$$\lambda_j^{(i)}, \qquad 1 \leqq i \leqq k,\ 1 \leqq j \leqq r_i,$$

and 0 *with multiplicity* $n - (r_1 + \cdots + r_k)$ *then the ranges of* N_i, $1 \leqq i \leqq k$, *are orthogonal to each other,* N *is normal and* $N_i N_j = 0$ *for* $i \neq j$.

PROOF. Let V_i be the range of N_i. Since

$$\operatorname{rank} N = \sum_{i=1}^{k} \operatorname{rank} N_i$$

we infer that the sum $U = V_1 + \cdots + V_k$ is direct. Let W be the orthogonal complement of U. Then W is contained in the null space of each operator N_i. Hence W is the null space of N.

Hence, without loss of generality, we can assume that $r_1 + \cdots + r_k = n$; i.e., that N is nonsingular.

Let $\boldsymbol{P}_i$ be the orthogonal projector in V with range V_i. Let $\boldsymbol{P}_{ji}: V_j \to V_i$ be the restriction of $\boldsymbol{P}_i$ and $\boldsymbol{M}_i: V_i \to V_i$ the restriction of $\boldsymbol{N}_i$. We fix a basis in each V_i. Since

$$\boldsymbol{N} = \sum_{i=1}^{k} \boldsymbol{N}_i = \sum_{i=1}^{k} \boldsymbol{N}_i \boldsymbol{P}_i = \sum_{i=1}^{k} \boldsymbol{M}_i \boldsymbol{P}_i$$

the matrix of $\boldsymbol{N}$ has the form

$$N = \begin{pmatrix} M_1 P_{11} & M_1 P_{21} \cdots \\ M_2 P_{12} & M_2 P_{22} \\ \vdots & \end{pmatrix}$$

where M_i, P_{ij} are matrices of $\boldsymbol{M}_i$, $\boldsymbol{P}_{ij}$.

The conditions on eigenvalues of $\boldsymbol{N}_i$ and $\boldsymbol{N}$ imply that

$$\det(\boldsymbol{N} - \lambda \boldsymbol{I}) = \prod_{i=1}^{k} \det(\boldsymbol{M}_i - \lambda \boldsymbol{I}_i).$$

For $\lambda = 0$ we get $\det(\boldsymbol{N}) = \prod_{i=1}^{k} \det(\boldsymbol{M}_i)$. Since

$$N = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix} \begin{pmatrix} P_{11} & P_{21} \cdots \\ P_{12} & P_{22} \cdots \\ \vdots & \end{pmatrix}$$

we infer that

$$\det \begin{pmatrix} P_{11} & P_{21} \cdots \\ P_{12} & P_{22} \cdots \\ \vdots & \vdots \end{pmatrix} = 1.$$

This is equivalent to

$$\det(\boldsymbol{P}_1 + \cdots + \boldsymbol{P}_k) = 1.$$

Theorem 2 implies that $V = V_1 + \cdots + V_k$ is an orthogonal direct sum.

The remaining assertions follow easily.

References

1. L. Brand, *On the product of singular symmetric matrices*, Proc. Amer. Math. Soc. **22** (1969), 377. MR **39** #4183.

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