ON THE INVERTIBILITY OF GENERAL WIENER-HOPF OPERATORS

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Abstract. Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}$ the set of bounded linear operators on $\mathcal{H}$, and $P$ an orthogonal projection on $\mathcal{H}$. Denote the range of $P$ by $R(P)$. Let $A$ belong to $\mathfrak{B}$. The general Wiener-Hopf operator associated with $A$ and $P$ is defined by $T_P(A) = PA | R(P)$, the vertical bar denoting restriction. Let $Q = I - P$. The purpose of this paper is to disprove the general conjecture that if $A$ is an invertible element of $\mathfrak{B}$, then the invertibility of $T_P(A)$ implies the invertibility of $T_Q(A)$. We also disprove the conjecture in an interesting special case.

1. Introduction. Let $\mathcal{H}$ be a separable Hilbert space, and $\mathfrak{B}$ the set of bounded linear operators on $\mathcal{H}$. Let $P$ be an orthogonal projection on $\mathcal{H}$. Denote its range by $R(P)$. Let $A$ belong to $\mathfrak{B}$. The general Wiener-Hopf operator associated with $A$ and $P$ is defined by $T_P(A) = PA | R(P)$, the vertical bar denoting restriction. Let $Q = I - P$. The purpose of this paper is to disprove the general conjecture that if $A$ is an invertible element of $\mathfrak{B}$, then the invertibility of $T_P(A)$ implies the invertibility of $T_Q(A)$. We also disprove the conjecture in an interesting special case.

In §2 we mention some special cases in which the conjecture has been proven true. We then exhibit a simple counterexample to prove that the conjecture is, in general, false.

In order to discuss §3 we introduce some additional terminology. Let $L^2(T; \mathcal{H})$ denote the space of equivalence classes of $\mathcal{H}$-valued, weakly measurable functions on the circle group $T$ which are square summable; $H^2(T; \mathcal{H})$ the subspace of $L^2(T; \mathcal{H})$ which consists of those elements whose Fourier coefficients vanish on the negative integers; and $K^2(T; \mathcal{H})$ the orthogonal complement of $H^2(T; \mathcal{H})$ in $L^2(T; \mathcal{H})$. $P$ and $Q$ are projections from $L^2(T; \mathcal{H})$ onto $H^2(T; \mathcal{H})$ and $K^2(T; \mathcal{H})$ respectively. All operators are assumed to be bounded and linear. Let $A(0)$ be any weakly measurable, essentially bounded, $\mathfrak{B}$-valued function on the circle group. We define an operator $A$ on $L^2(T; \mathcal{H})$ by
\[(Af)(\theta) = A(\theta)f(\theta), \quad f \in L^2(T; \mathbb{S}), \quad \theta \in T,\]

and two Wiener-Hopf operators by
\[T_P(A) = PA \mid H^2(T; \mathbb{S}), \quad T_Q(A) = QA \mid K^2(T; \mathbb{S}).\]

As usual, \(\mathbb{C}\) denotes the complex numbers. Finally, suppose that the operator \(A\) is invertible.

Devinatz and Shinbrot [1] have shown that if \(H\) is one-dimensional, then \(T_P(A)\) being invertible implies \(T_Q(A)\) is invertible. They remark: "Whether or not this result persists when the dimension of \(\mathbb{S}\) is greater than 1 is not known." (The remark follows proof of Corollary 8, §6 in [1].) We shall prove that the result does not by exhibiting a counterexample when the dimension of \(\mathbb{S}\) is two (i.e. \(\mathbb{S} = \mathbb{C} \times \mathbb{C}\)). The counterexample is of the form \(U^*B\) \(U\) where \(U\) is a certain unitary operator on \(L^2(T; \mathbb{C} \times \mathbb{C})\) mapping \(H^2(T; \mathbb{C} \times \mathbb{C})\) onto \(L^2(T; \mathbb{C}) \times \{0\}\) and \(K^2(T; \mathbb{C} \times \mathbb{C})\) onto \(\{0\} \times L^2(T; \mathbb{C})\), and \(B\) is an invertible operator on \(L^2(T; \mathbb{C} \times \mathbb{C})\) patterned after the counterexample of §2. This counterexample may be generalized to disprove the result when the dimension of \(\mathbb{S}\) is any integer greater than one.

2. The first counterexample. Devinatz and Shinbrot have proven that for a unitary operator \(U\), the invertibility of \(T_P(U)\) is equivalent to that of \(T_Q(U)\) for any orthogonal projection \(P\) on \(\mathbb{S}\) where \(Q = I - P\) (Corollary 1, §2 in [1]). They have also shown that if \(A\) is any operator with a strongly positive real part \((\text{Re}(Ax, x) \geq \delta(x, x), x \in \mathbb{S}, \delta \text{ positive})\), then \(T_P(A)\) is invertible for every orthogonal projection \(P\). In particular, it is invertible for \(Q = I - P\) (Lemma 2, §2 in [1]). Pellegrini proved that for an operator \(A\), \(T_P(A)\) is invertible for every orthogonal projection if and only if there exists a \(\theta\) between 0 and \(2\pi\) such that \(e^{i\theta}A\) has strongly positive real part (Theorem 1.2.10 in [2]).

The following simple counterexample illustrates that the invertibility of \(T_P(A)\) does not guarantee that of \(T_Q(A)\), even if \(A\) is invertible.

**Counterexample 1.** Take \(\mathbb{S} = \mathbb{C} \times \mathbb{C}\), \(P\) the projection from \(\mathbb{S}\) onto \(\mathbb{C} \times \{0\}\), and \(Q = I - P\). Let \(A\) be the matrix \(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\). The determinant of \(A\) is \(-1\) so that \(A\) is invertible. It is easily seen that \(T_P(A) = I | \mathbb{C} \times \{0\}\) and \(T_Q(A) = 0 | \{0\} \times \mathbb{C}\).

3. The second counterexample. In this section \(P\) is the projection from \(L^2(T; \mathbb{C} \times \mathbb{C})\) onto \(H^2(T; \mathbb{C} \times \mathbb{C})\) and \(Q = I - P\); \(P'\) is the projection from \(L^2(T; \mathbb{C} \times \mathbb{C})\) onto \(L^2(T, \mathbb{C}) \times \{0\}\) and \(Q' = I - P'\). We shall construct an invertible operator \(A\) on \(L^2(T; \mathbb{C} \times \mathbb{C})\) such that \(T_P(A)\) is invertible but \(T_Q(A)\) is not.
Let \([ , ]\) denote the inner product in both \(\mathbb{C} \times \mathbb{C}\) and \(\mathbb{C}\); \(\langle , \rangle\) the inner product in both \(L^2(T; \mathbb{C} \times \mathbb{C})\) and \(L^2(T; \mathbb{C})\). If \(f(\theta) \in L^2(T; \mathbb{C} \times \mathbb{C})\), then \(f\) is easily seen to be of the form \((f_1(\theta), f_2(\theta))\) where \(f_1, f_2 \in L^2(T; \mathbb{C})\). Let \(g(\theta) = (g_1(\theta), g_2(\theta))\) be another element of \(L^2(T; \mathbb{C} \times \mathbb{C})\).

The topology on \(L^2(T; \mathbb{C} \times \mathbb{C})\) is determined by the inner product:

\[
(3.1) \quad \langle f, g \rangle = \int_{\theta \in T} [f(\theta), g(\theta)]d\theta = \int_{\theta \in T} [(f_1(\theta), f_2(\theta)), (g_1(\theta), g_2(\theta))]d\theta.
\]

We define inner product on \(\mathbb{C} \times \mathbb{C}\) to be the sum of the respective inner products. Thus, (3.1) becomes

\[
(3.2) \quad \langle f, g \rangle = \int_{\theta \in T} [f_1(\theta), g_1(\theta)]d\theta + \int_{\theta \in T} [f_2(\theta), g_2(\theta)]d\theta
= \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle.
\]

This indicates that we may identify \(L^2(T; \mathbb{C} \times \mathbb{C})\) with \(L^2(T; \mathbb{C}) \times L^2(T; \mathbb{C})\) where the inner product in the latter space is given by the right-hand side of (3.2). Under this identification, \(H^2(T; \mathbb{C} \times \mathbb{C})\) and \(K^2(T; \mathbb{C} \times \mathbb{C})\) become \(H^2(T; \mathbb{C}) \times H^2(T; \mathbb{C})\) and \(K^2(T; \mathbb{C}) \times K^2(T; \mathbb{C})\) respectively. We make this identification freely throughout the remainder of the paper.

The Plancherel theorem says that the Fourier transform, \(F\), is an isometry from \(L^2(T; \mathbb{C})\) onto \(l^2\), the square-summable, \(\mathbb{C}\)-valued sequences on the integers. The adjoint of \(F\), \(F^*\), is easily seen to equal \(F^{-1}\). We make use of these facts in the following lemma.

**Lemma 1.** There exists a unitary operator \(U\) on \(L^2(T; \mathbb{C} \times \mathbb{C})\) such that

(i) \(U = U^* = U^{-1}\),

(ii) \(U(H^2(T; \mathbb{C} \times \mathbb{C})) = L^2(T; \mathbb{C}) \times \{0\}\),

(iii) \(U(K^2(T; \mathbb{C} \times \mathbb{C})) = \{0\} \times L^2(T; \mathbb{C})\).

**Proof.** For \((f_n), (g_n) \in l^2\), define a map \(G: l^2 \times l^2 \to l^2 \times l^2\) by

\[
G((f_n), (g_n)) = ((h_n), (k_n))
\]

where

\[
h_n = f_n, \quad n \geq 0,
= g_{-n-1}, \quad n < 0,
\]

\[
k_n = f_{-n-1}, \quad n \geq 0,
= g_n, \quad n < 0.
\]
\[ \| G((f_n), (g_n)) \|_2^2 = \sum f_n \bar{f}_n + \sum g_{-n-1} \bar{g}_{-n-1} + \sum f_{-n} \bar{f}_{-n-1} + \sum g_n \bar{g}_n = \| (f_n) \|_2^2 + \| (g_n) \|_2^2. \]

The inner product in \( l^2 \times l^2 \) is taken to be the sum of the respective inner products in \( l^2 \), so that \( \| (f_n), (g_n) \|_2^2 = \| (f_n) \|_2^2 + \| (g_n) \|_2^2 \). Observe that \( G \) is linear. Hence \( G \) is an isometry. Note that \( G^2 = I \), so that \( G = G^{-1} \). Let \( h^2 \times h^2 (k^2 \times k^2) \) denote the subspace of \( l^2 \times l^2 \) of series whose terms vanish for negative (positive) indices. Observe that \( G \) maps \( h^2 \times h^2 \) onto \( l^2 \times \{0\} \) and \( k^2 \times k^2 \) onto \( \{0\} \times l^2 \).

Now we define a map \( H: L^2(T; \mathbb{C}) \times L^2(T; \mathbb{C}) \rightarrow l^2 \times l^2 \) via the Fourier transform, \( F \), as follows

\[ (H(f, g))_n = \langle Ff, Fg \rangle, \quad (f, g) \in L^2(T; \mathbb{C}) \times L^2(T; \mathbb{C}), \]

\[ \| H(f, g) \|_2^2 = \| Ff \|_2^2 + \| Fg \|_2^2 = \| f \|_2^2 + \| g \|_2^2. \]

Note that \( H \) is linear because \( F \) is linear. Hence \( H \) is an isometry. It takes \( H^2(T; \mathbb{C}) \times H^2(T; \mathbb{C}) \) onto \( h^2 \times h^2 \), \( K^2(T; \mathbb{C}) \times K^2(T; \mathbb{C}) \) onto \( K^2 \times K^2 \), \( L^2(T; \mathbb{C}) \times \{0\} \) onto \( l^2 \times \{0\} \), and \( \{0\} \times L^2(T; \mathbb{C}) \) onto \( \{0\} \times l^2 \). It is easily seen that \( H^* = H^{-1} \).

Finally, we set \( U = H^{-1}GH \). If we make the identification of \( L^2(T; \mathbb{C} \times \mathbb{C}) \) with \( L^2(T; \mathbb{C}) \times L^2(T; \mathbb{C}) \) mentioned earlier, then \( U \) may be considered as an operator taking \( L^2(T; \mathbb{C} \times \mathbb{C}) \) onto itself. Now (ii) and (iii) follow immediately from the above mentioned properties of \( G \) and \( H \). We prove (i):

\[ U^* = H^* G^* (H^{-1})^* = H^{-1} GH = U, \quad U^{-1} = H^{-1} G^{-1} H = H^{-1} GH = U. \]

The following operator is central to the forthcoming counterexample. Its construction is patterned after that of Counterexample 1. Let

\[ B(\theta) = \begin{pmatrix} b_{11}(\theta) & b_{12}(\theta) \\ b_{21}(\theta) & b_{22}(\theta) \end{pmatrix} \]

(3.3)

where \( \begin{cases} b_{11}(\theta) = b_{12}(\theta) = b_{21}(\theta) = 1, \\ b_{22}(\theta) = 0, \end{cases} \quad \theta \in T. \)

Define

\[ (Bf)(\theta) = B(\theta)f(\theta), \quad f \in L^2(T; \mathbb{C} \times \mathbb{C}), \quad \theta \in T. \]

The properties of \( B \) which will be of interest to us are contained in the following lemma.

**Lemma 2.** \( B \) is an invertible operator on \( L^2(T; \mathbb{C} \times \mathbb{C}) \) such that

(i) \( T_{P'}(B) = I \cdot L^2(T; \mathbb{C}) \times \{0\} \),

(ii) \( T_{Q'}(B) = 0 \cdot \{0\} \times L^2(T; \mathbb{C}). \)
Proof. For any \( f \in L^2(T; \mathbb{C} \times \mathbb{C}) \) we have shown that \( f = (f_1, f_2) \) where \( f_1, f_2 \in L^2(T; \mathbb{C}) \). By (3.3), \( B(f_1, f_2) = (f_1 + f_2, f_1) \). Hence (i) and (ii) follow immediately. It is easily seen that \( B^{-1}(f_1, f_2) = (f_2, f_1 - f_2) \). The fact that \( B \) and \( B^{-1} \) are bounded and linear is also easily shown.

The following simple lemma is proven in a general context.

Lemma 3. Let \( U \) be a unitary operator on a Hilbert space \( \mathcal{H} \). Suppose \( P \) and \( P' \) are two orthogonal projections on \( \mathcal{H} \) such that \( U(R(P)) = R(P') \) and \( U(R(Q)) = R(Q') \) (\( Q = I - P, \ Q' = I - P' \)). Let \( B \) belong to \( \mathfrak{Y} \).

(i) \( TP(U^*BU) = U^*TP'(B)U \mid R(P) \),

(ii) \( TQ(U^*BU) = U^*TQ'(B)U \mid R(Q) \).

Proof. First we note that \( (U^*P'U)^2 = U^*P'U, (U^*P'U)^* = U^*P'U, \) and \( R(U^*P'U) = R(P) \). Hence \( P = U^*P'U \). The rest is easy:

\[
U^*TP'(B)U \mid R(P) = U^*(P'UU*B \mid R(P'))U \mid R(P) = P(U^*BU) \mid R(P) = TP(U^*BU).
\]

The proof for \( TQ(U^*BU) \) is identical.

We are now prepared to exhibit the following counterexample.

Counterexample 2. There exists an invertible operator \( A \) on \( L^2(T; \mathbb{C} \times \mathbb{C}) \) such that \( TP(A) = I \mid H^2(T; \mathbb{C} \times \mathbb{C}) \) and \( TQ(A) = 0 \mid K^2(T; \mathbb{C} \times \mathbb{C}) \).

Proof. Let \( U \) be the operator of Lemma 1; \( B \), that of Lemma 2; and set \( A = U^*BU \). By Lemmas 1 and 2, \( A \) is an invertible operator on \( L^2(T; \mathbb{C} \times \mathbb{C}) \). Moreover, Lemma 3 says that

\[
U^*TP'(B)U \mid H^2(T; \mathbb{C} \times \mathbb{C}) = TP(U^*BU).
\]

But \( TP'(B) = I \mid L^2(T; \mathbb{C} \times \{0\}) \) by (i) of Lemma 2. Thus, using (i) and (ii) of Lemma 1, we get that

\[
TP(A) = TP(U^*BU) = I \mid H^2(T; \mathbb{C} \times \mathbb{C}).
\]

Similarly, one sees that

\[
TQ(A) = TQ(U^*BU) = U^*TQ'(B)U \mid K^2(T; \mathbb{C} \times \mathbb{C}) = 0 \mid K^2(T; \mathbb{C} \times \mathbb{C}).
\]

References


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