TOPOLOGICAL CHARACTERIZATION OF PSEUDO-$\mathfrak{N}$-COMPACT SPACES

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Abstract. Pseudo-$\mathfrak{N}$-compact spaces, usually defined by cardinal bounds on uniform coverings of the fine uniformity, are characterized by cardinal bounds on locally finite open (resp. closed) coverings. This yields a new characterization of pseudo-compact spaces as well as of G. Aquaro's paralindelöfian spaces.

According to [4, p. 135], a pseudo-$\mathfrak{N}$-compact space, $\mathfrak{N}$ an infinite cardinal, is a uniformizable space whose fine uniformity has covering character $\leq \mathfrak{N}$—which means that every uniform covering of the fine uniformity (= finest compatible uniformity) has a uniform refinement of cardinality $<\mathfrak{N}$. For $\mathfrak{N} = \mathfrak{N}_0$, we have the well-known pseudocompact spaces; for $\mathfrak{N} = \mathfrak{N}_1$, we have the paralindelöfian spaces of [1]. Although many topological characterizations of pseudocompact spaces are well known, there is no topological characterization of pseudo-$\mathfrak{N}$-compact spaces with $\mathfrak{N} > \mathfrak{N}_0$, since in [1] and [4, Chapter vii] they are investigated only from the viewpoint of uniform spaces (roughly speaking, [1, Proposition 3] characterizes paralindelöfian spaces by $\mathfrak{U}$-reducible locally finite open coverings, but this characterization—a part of the very complicated notion of "$\mathfrak{U}$-reducible"—is not purely topological since $\mathfrak{U}$-reducible locally finite open coverings are exactly the elements of a base for the fine uniformity).

In this note, pseudo-$\mathfrak{N}$-compact spaces are characterized by cardinal bounds on locally finite open or closed coverings. This should be new also for pseudocompact spaces since no cardinal bounds are assumed, a priori, on coverings.

Terminology. Our terminology is based on [4], with the sole exception that our spaces are not assumed, a priori, to be Hausdorff.

Theorem. For a uniformizable space $X$ the following statements are pairwise equivalent:

1. $X$ is pseudo-$\mathfrak{N}$-compact.
2. Every locally finite disjoint collection of nonempty open sets of $X$ has cardinality $<\mathfrak{N}$.

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(3) Every locally finite closed covering of $X$ has a subcovering of cardinality $<\mathbb{N}$.

(4) Every locally finite open covering $\mathcal{U}$ of $X$ has a subcollection $\mathcal{V}$ such that $\text{Card}(\mathcal{V}) < \mathbb{N}$ and $X = \bigcup_{V \in \mathcal{V}} V$.

**Proof.** (1) $\rightarrow$ (2). Let $\mathcal{U}$ be a locally finite disjoint collection of non-empty open sets of $X$. In each $U \in \mathcal{U}$ choose a point $x_U$. Since $X$ is uniformizable, for every $U \in \mathcal{U}$ there is a continuous map $f_U : X \to [0,1]$ which is 1 on $x_U$ and 0 on $X \setminus U$. Let $e_U = (y_V)_{V \in \mathcal{U}}$ be the element of the Hilbert space $l_2(\mathcal{U})$—see [3, Chapter ix, §8] for its definition—defined by $y_U = 1$ and $y_V = 0$ for $V \neq U$. Since $\mathcal{U}$ is locally finite,

$$f(x) = \sum_{U \in \mathcal{U}} f_U(x)e_U$$

defines a continuous map $f : X \to l_2(\mathcal{U})$. Let $\mu$ be the fine uniformity of $X$. By a well-known theorem, [4, i.21], $f : \mu X \to l_2(\mathcal{U})$ is uniformly continuous. Therefore there is a uniform cover $\mathcal{V} \subseteq \mu$ such that

$$\text{diam}(f(V)) < 1/2 \quad (V \in \mathcal{V}).$$

By (1), $\mathcal{V}$ has a uniform refinement $\mathcal{W}$ of cardinality $<\mathbb{N}$. For every $U \in \mathcal{U}$ there is $W_U \subseteq \mathcal{W}$ such that $x_U \in W_U$. Let us show that $U' \neq U''$ implies $W_{U'} \neq W_{U''}$. For, $W_{U'} = W_{U''}$ implies $x_{U'}$ and $x_{U''} \in W_{U'}$, so that

$$\|f(x_{U'}) - f(x_{U''})\| < 1/2.$$

But this is impossible since $U_0 \neq U$ implies $f_{U_0}(x_U) = 0$, so that $f(x_{U'}) = e_{U'}$, and $\|e_{U'} - e_{U''}\| = \sqrt{2}$. Hence $U \rightarrow W_U$ is a 1-1 map $\mathcal{U} \rightarrow \mathcal{W}$, so that $\text{Card}(\mathcal{U}) \leq \text{Card}(\mathcal{W}) < \mathbb{N}$.

(2) $\rightarrow$ (3). Let $(C_\alpha)_\alpha$ be a locally finite closed covering of $X$. By [3, p. 177, Exercises 7 and 6], there is a disjoint family $(U_\alpha)_\alpha$ of open sets of $X$ such that $\bigcup U_\alpha \subseteq C_\alpha$ for every $\alpha$, and $X = \bigcup U_\alpha$. (We recall briefly the construction: Let $\leq$ be a well-order for the indexing set. One may show inductively that

$$\{\text{Int}(C_\beta) \setminus \beta \leq \alpha\} \cup \{C_\beta \mid \beta > \alpha\}$$

is a covering of $X$ for every $\alpha$. Consequently it is easily seen that the sets

$$U_\alpha = \text{Int}(C_\alpha) \setminus \bigcup_{\rho < \alpha} \text{Int}(C_\rho)$$

form the desired family.) Then obviously (3) follows from (2).

(3) $\rightarrow$ (4). Because $\{U \mid U \in \mathcal{U}\}$ is a locally finite closed covering of $X$. [January]
(4)→(1). Let $\mathcal{U}$ be a uniform cover of the fine uniformity $\mu$ of $X$. It follows from [4, i.14] that $\mathcal{U}$ has a closed uniform refinement $\mathcal{V}$. By [4, vii. 4 and i.19] or, alternatively, [2, §3], $\mathcal{V}$ has an open locally finite uniform refinement $\mathcal{W}$. By (4), there is $\mathcal{W}_0 \subseteq \mathcal{W}$ such that $X = \bigcup_{W \in \mathcal{W}_0} W$ and $\text{Card}((\mathcal{W}_0)) < \mathbb{K}$, so that $\mathcal{U}$ has a refinement of cardinality $< \mathbb{K}$. Hence $X$ is pseudo-$\mathbb{K}$-compact by [4, ii.33]. q.e.d.

As pointed out by [1], the one-point compactification of a discrete uncountable space cannot have Souslin property, so that local finiteness cannot be relaxed in the above theorem. Since any uniform cover of the fine uniformity is refined by a $\sigma$-uniformly discrete covering by [2, §3] or, alternatively, [4, vii.4 and i.19], we may change "locally finite" with "discrete" in statement (2) of the theorem.

The following result is well known for pseudocompact spaces.

**Corollary.** If $U$ is an open subset of a pseudo-$\mathbb{K}$-compact space, then $\overline{U}$ is pseudo-$\mathbb{K}$-compact.

**Proof.** Let $(C_\alpha)_{\alpha \in \Lambda}$ be a closed covering of $\overline{U}$ which is locally finite in the subspace $\overline{U}$ of $X$. Clearly $\{C_\alpha | \alpha \in \Lambda \} \cup \{X \setminus U\}$ is a locally finite closed covering of $X$, so that it has a subcovering of cardinality $< \mathbb{K}$ by the theorem. Therefore there is $A' \subseteq A$ such that $\text{Card}(A') < \mathbb{K}$ and $U \subseteq \bigcup_{\alpha \in A'} C_\alpha$. Since $(C_\alpha)_\alpha$ is locally finite and closed in $\overline{U}$,

$$\overline{U} \subseteq \bigcup_{\alpha \in A'} C_\alpha = \bigcup_{\alpha \in A'} C_\alpha.$$ q.e.d.

Define $X$ locally pseudo-$\mathbb{K}$-compact iff $X$ is uniformizable and every point of $X$ has a neighborhood which is a pseudo-$\mathbb{K}$-compact subspace of $X$. From the preceding corollary it follows that in a locally pseudo-$\mathbb{K}$-compact space every point has a neighborhood base made of pseudo-$\mathbb{K}$-compact subspaces. Let $X$ be a locally pseudo-$\mathbb{K}$-compact space, $\emptyset$ the collection of all its pseudo-$\mathbb{K}$-compact subspaces, $x$ a point not in $X$, and $Y = X \cup \{x\}$. Let $\tau_0$ be the topology on $Y$ having as basis

$$\{U \subseteq X | U \text{ is open} \} \cup \{ Y \setminus P | P \in \emptyset \}.$$ Let $\tau$ be the weak topology induced on $Y$ by all continuous functions $(Y, \tau_0) \rightarrow \mathbb{R}$. One can derive from the theorem that $(Y, \tau)$ is pseudo-$\mathbb{K}$-compact. Moreover, $id : X \rightarrow (Y, \tau)$ is a homeomorphism into. This space $(Y, \tau)$ may be looked at the one-point pseudo-$\mathbb{K}$-compactification of $X$. The key properties of one-point pseudo-$\mathbb{K}$-compactifications are like those of one-point compactifications, with "pseudo-$\mathbb{K}$-compact" in place of "compact".
Finally, note that every open subset of an arbitrary product of uniformizable spaces, each with density character $\leq \aleph_\alpha$, is pseudo-$\aleph_{\alpha+1}$-compact. For, every uniform covering is realized by a continuous map into a metric space, and such a map may be factored (as one may check by repeating the proof of Gleason factorization theorem [4, vii.23]) as $g \circ \pi_c$ with $\pi_c$ a countable projection and $g$ continuous.

References


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