ON THE STRUCTURE OF A FINITE SOLVABLE $K$-GROUP

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Abstract. In this note we investigate the structure of a finite solvable $K$-group. It is proved that a finite group $G$ is a solvable $K$-group if and only if $G$ is a subdirect product of a finite collection of solvable $K$-groups $H_i$ such that each $H_i$ is isomorphic to a subgroup of $G$, and each $H_i$ possesses a unique minimal normal subgroup.

This note eliminates several errors in [1]. The statement and the proof of Theorem 1 need correction. The statement of Theorem 1 should read "... collection of solvable $K$-groups ..." rather than "... collection of $K$-groups ...," while its proof is invalid. In this note we will consider only finite groups with the notation and terminology in [1] assumed to be known.

The following results by Zacher [4] will be useful:

(1) A finite solvable group $G$ is a $K$-group if and only if $G$ contains a series of normal subgroups $1 = N_0 < N_1 < \cdots < N_r = G$ such that each $N_{i+1}/N_i$ is a maximal normal nilpotent subgroup of $G/N_i$ and $\Phi(G/N_i) = 1$ for $i = 0, 1, \cdots, r - 1$.

(2) Each homomorphic image of a solvable $K$-group is a $K$-group.

Theorem 1. A group $G$ is a solvable $K$-group if and only if $G$ is a subdirect product of a finite collection of solvable $K$-groups $H_i$ such that each $H_i$ is isomorphic to a subgroup of $G$, and each $H_i$ possesses a unique minimal normal subgroup.

Proof. Considering a solvable group $G$, we know that $G$ is a subdirect product of a finite collection of solvable groups $H_i$ such that for all $i$, $H_i$ has a unique minimal normal subgroup. Adding the condition that $G$ is a $K$-group, we denote by $A$ the kernel of the projection of $G$ onto $H_i$ for some $i$. In particular, using the normality of $A$ in the $K$-group $G$, it must follow that there exists a subgroup $B$ of $G$ such that $G = AB$ with $A \cap B = 1$. In particular, we observe that $B \cong H_i$, i.e. the...
direct factors in the subdirect product are isomorphic to subgroups of \( G \). Applying (2) we note that \( H_i \) is a \( K \)-group. Thus we have established the necessity of the conditions of the theorem.

The sufficiency of the conditions of the above theorem follows by induction on the order of \( G \). Suppose the result holds for all groups of order less than \( G \). We denote the projection of \( G \) on \( H_i \) by \( \pi_i \), and the kernel of \( \pi_i \) by \( A_i \), and note that \( \cap_{i \in I} A_i = 1 \). Since \( H_i \) is a \( K \)-group, \( \Phi(H_i) = 1 \), and since \( \Phi(G)\pi_i \leq \Phi(H_i) \) for all \( i \in I \), then \( \Phi(G) \leq \cap_{i \in I} A_i = 1 \). Thus for each subgroup \( N \) normal in \( G \), \( \Phi(N) = 1 \) (see [2]). But this implies that the Fitting group \( F(G) \) is elementary abelian, and that \( G \) splits over \( F(G) \). That is, there exists a subgroup \( C \) such that \( G = F(G) \cdot C \) with \( F(G) \cap C = 1 \). Note that \( G \) is clearly solvable, and we are assuming \( G \neq 1 \); thus \( \Phi(G) = 1 < F(G) \). Thus the order of \( C \) is less than the order of \( G \). Since each \( H_i \) is a solvable \( K \)-group, \( F(H_i) \) is completely reducible. Thus \( F(H_i) \) must be the unique minimal normal subgroup of \( H_i \) for each \( i \). Now we note that \( H_i = (F(G)\pi_i)(C\pi_i) \).

If \( F(G)\pi_i \cap C\pi_i = 1 \), then \( H_i / F(G)\pi_i \cong C\pi_i \). Thus \( C\pi_i \) is a solvable \( K \)-group. If \( F(G)\pi_i \cap C\pi_i = B_i \neq 1 \), then \( B_i \leq F(G)\pi_i \leq F(H_i) \), and \( B_i \neq 1 \) implies \( B_i = F(H_i) \). Furthermore since \( F(G)\pi_i \leq F(H_i) \), it follows that \( F(G)\pi_i \leq C\pi_i = H_i \). (I.e. since \( F(G)\pi_i \leq F(H_i) = B_i \), and \( B_i \leq C\pi_i \).

Thus \( F(G)\pi_i \leq C\pi_i \). Furthermore since \( H_i = (F(G)\pi_i)(C\pi_i) \), \( g \in H_i \) implies \( g = fk \) where \( f \in F(G)\pi_i \), \( k \in C\pi_i \). Thus \( f \in F(H_i) \), \( k \in C\pi_i \) implies \( f \in B_i \leq C\pi_i \), and \( k \in C\pi_i \), and all this implies \( f = g \in C\pi_i \). Again \( C\pi_i \) is a solvable \( K \)-group. Since for each \( i \in I \), \( C\pi_i \) is a solvable \( K \)-group, it follows that each \( C\pi_i \) is a subdirect product of solvable \( K \)-groups each having precisely one minimal normal subgroup. If we combine the \( H_i \) for which \( C\pi_i = H_i \) together with the direct factors in the subdirect product associated with those \( C\pi_i \) for which \( C\pi_i \neq H_i \), a direct product can be formed for which \( C \) is a subdirect product of solvable \( K \)-groups containing precisely one minimal normal subgroup where we use composition of mappings where necessary. By the inductive hypothesis \( C \) is a solvable \( K \)-group. But this implies that in \( G = F(G) \cdot C \) there exists a series of subgroups \( 1 = N_0 < N_1 < \cdots < N_r = G \) such that \( N_i / N_{i-1} \) is the maximal normal nilpotent subgroup of \( G / N_{i-1} \) and \( \Phi(G / N_{i-1}) = 1 \). Now using Zacher's result (1) above, we note that \( G \) is a \( K \)-group.

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**Bibliography**


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