PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, January 1971

ON THE STRUCTURE OF A FINITE SOLVABLE K-GROUP

MARSHALL KOTZEN

ABSTRACT. In this note we investigate the structure of a finite solvable K-group. It is proved that a finite group G is a solvable K-group if and only if G is a subdirect product of a finite collection of solvable K-groups H_i such that each H_i is isomorphic to a subgroup of G, and each H_i possesses a unique minimal normal subgroup.

This note eliminates several errors in [1]. The statement and the proof of Theorem 1 need correction. The statement of Theorem 1 should read " $\cdots$ collection of solvable K-groups $\cdots$ " rather than " $\cdots$ collection of K-groups $\cdots$," while its proof is invalid. In this note we will consider only finite groups with the notation and terminology in [1] assumed to be known.

The following results by Zacher [4] will be useful:

(1) A finite solvable group G is a K-group if and only if G contains a series of normal subgroups $1=N_0<N_1<\cdots<N_r=G$ such that each N_{i+1}/N_i is a maximal normal nilpotent subgroup of G/N_i and $\Phi(G/N_i)=1$ for $i=0, 1, \cdots, r-1$.

(2) Each homomorphic image of a solvable K-group is a K-group.

THEOREM 1. *A group G is a solvable K-group if and only if G is a subdirect product of a finite collection of solvable K-groups H_i such that each H_i is isomorphic to a subgroup of G, and each H_i possesses a unique minimal normal subgroup.*

PROOF. Considering a solvable group G, we know that G is a subdirect product of a finite collection of solvable groups H_i such that for all i, H_i has a unique minimal normal subgroup. Adding the condition that G is a K-group, we denote by A the kernel of the projection of G onto H_i for some i. In particular, using the normality of A in the K-group G, it must follow that there exists a subgroup B of G such that $G=AB$ with $A\cap B=1$. In particular, we observe that $B\cong H_i$, i.e. the

Received by the editors November 18, 1969.

AMS 1969 *subject classifications*. Primary 2000; Secondary 2025, 2027, 2052.

Key words and phrases. Solvable group, K-group, subdirect product, unique minimal normal subgroup, series, maximal normal nilpotent subgroup, Frattini subgroup, Fitting group, elementary abelian, G splits over $F(G)$, completely reducible, direct product.

direct factors in the subdirect product are isomorphic to subgroups of G. Applying (2) we note that H_i is a K-group. Thus we have established the necessity of the conditions of the theorem.

The sufficiency of the conditions of the above theorem follows by induction on the order of G. Suppose the result holds for all groups of order less than G. We denote the projection of G on H_i by π_i and the kernel of π_i by A_i, and note that $\bigcap_{i\in I} A_i = 1$. Since H_i is a K-group, $\Phi(H_i)=1$, and since $\Phi(G)\pi_i \leqq \Phi(H_i)$ for all $i\in I$, then $\Phi(G)\leqq \bigcap_{i\in I} A_i = 1$. Thus for each subgroup N normal in G, $\Phi(N)=1$ (see [2]). But this implies that the Fitting group $F(G)$ is elementary abelian, and that G splits over $F(G)$. That is, there exists a subgroup C such that $G=F(G)\cdot C$ with $F(G)\cap C=1$. Note that G is clearly solvable, and we are assuming $G\neq 1$; thus $\Phi(G)=1<F(G)$. Thus the order of C is less than the order of G. Since each H_i is a solvable K-group, $F(H_i)$ is completely reducible. Thus $F(H_i)$ must be the unique minimal normal subgroup of H_i for each i. Now we note that $H_i=(F(G)\pi_i)(C\pi_i)$. If $F(G)\pi_i\cap C\pi_i=1$, then $H_i/F(G)\pi_i\cong C\pi_i$. Thus $C\pi_i$ is a solvable K-group. If $F(G)\pi_i\cap C\pi_i=B_i\neq 1$, then $B_i\leqq F(G)\pi_i\leqq F(H_i)$, and $B_i\neq 1$ implies $B_i=F(H_i)$. Furthermore since $F(G)\pi_i\leqq F(H_i)$, it follows that $F(G)\pi_i\leqq C\pi_i=H_i$. (I.e. since $F(G)\pi_i\leqq F(H_i)=B_i$, and $B_i\leqq C\pi_i$; $F(G)\pi_i\leqq C\pi_i$. Furthermore since $H_i=(F(G)\pi_i)(C\pi_i)$, $g\in H_i$ implies $g=fk$ where $f\in F(G)\pi_i$, $k\in C\pi_i$. Thus $f\in F(H_i)$, $k\in C\pi_i$ implies $f\in B_i\leqq C\pi_i$, and $k\in C\pi_i$ and all this implies $fk=g\in C\pi_i$.) Again $C\pi_i$ is a solvable K-group. Since for each $i\in I$, $C\pi_i$ is a solvable K-group, it follows that each $C\pi_i$ is a subdirect product of solvable K-groups each having precisely one minimal normal subgroup. If we combine the H_i for which $C\pi_i=H_i$ together with the direct factors in the subdirect product associated with those $C\pi_i$ for which $C\pi_i\neq H_i$, a direct product can be formed for which C is a subdirect product of solvable K-groups containing precisely one minimal normal subgroup where we use composition of mappings where necessary. By the inductive hypothesis C is a solvable K-group. But this implies that in $G=F(G)\cdot C$ there exists a series of subgroups $1=N_0<N_1<\cdots<N_r=G$ such that N_i/N_{i-1} is the maximal normal nilpotent subgroup of G/N_{i-1} and $\Phi(G/N_{i-1})=1$. Now using Zacher's result (1) above, we note that G is a K-group.

I thank Professor Bechtell for suggesting this paper to me.

Bibliography

1. H. Bechtell, *A note on finite solvable K-groups*, Proc. Amer. Math. Soc. **17** (1966), 1447–1450. MR **35** #4302.

2. W. Gaschütz, *Über die Φ-Untergruppe endlichen Gruppen*, Math. Z. **58** (1953), 160–170. MR **15**, 285.

3. M. Suzuki, *Structure of a group and the structure of its lattice of subgroups*, Ergebnisse der Math., Heft 10, Springer-Verlag, Berlin, 1956. MR **18**, 715.

4. G. Zacher, *Caratterizzazione dei gruppi risolubili d'ordine finite complementati*, Rend. Sem. Mat. Univ. Padova **22** (1953), 113–122. MR **15**, 286; MR **16**, 1336.

STATE COLLEGE AT WORCESTER, WORCESTER, MASSACHUSETTS 01602