IDEAL PRESERVING AUTOMORPHISMS OF POSTLIMINARY $C^*$-ALGEBRAS

GEORGE A. ELLIOTT

Abstract. E. C. Lance has shown that an automorphism of a separable postliminary $C^*$-algebra which preserves closed two-sided ideals is universally weakly inner. This note extends Lance's result to the nonseparable case.

1. Let $A$ be a $C^*$-algebra, and let $\alpha$ be an automorphism of $A$. If $\pi$ is a representation of $A$, $\alpha$ is weakly inner with respect to $\pi$ if there exists an element $U$ of the weak closure of $\pi(A)$ such that $\pi(\alpha(x)) = U\pi(x)U^*$ for all $x \in A$. Then necessarily $UU^*$ is equal to $U^*U$ and is the largest projection in the weak closure of $\pi(A)$; hence $\alpha(Ker \pi) = Ker \pi$. $\alpha$ is universally weakly inner if $\alpha$ is weakly inner with respect to every representation. This implies that $\alpha(J) = J$ for every closed two-sided ideal $J$ of $A$.

Theorem. Suppose that $A$ is a postliminary $C^*$-algebra ([1, §4]). Suppose that $\alpha$ is an automorphism of $A$ such that $\alpha(J) = J$ for every closed two-sided ideal $J$ of $A$. Then $\alpha$ is universally weakly inner.

With the additional hypothesis that $A$ be separable, this result is due to E. C. Lance ([3, Theorem 2]). In [3] Lance used the measure theoretic decomposition of a multiplicity-free representation of a separable $C^*$-algebra into irreducible representations ([1, §8]). The technique that we shall use here is the topological decomposition of a postliminary $C^*$-algebra into elementary $C^*$-algebras ([1, §10]). This in general is only possible locally; that is, after passing to a (nonzero) closed two-sided ideal. Accordingly, we shall first prove a local result (Lemma 2) and then deduce Theorem 1 with the aid of Zorn's lemma.

2. Lemma. Let $A$ and $\alpha$ be as in Theorem 1, and let $J \neq A$ be a closed two-sided ideal of $A$. Then there exists a closed two-sided ideal $J_0 \subset J$ such that the automorphism $x + J \mapsto \alpha(x) + J$ of $J_0/J$ is universally weakly inner.
Proof. Passing to the quotient of $A$ by $J$, we may suppose that $J = 0$.

Passing to a closed two-sided ideal of $A$, we may suppose that $A$ is the $C^*$-algebra defined by the continuous field of elementary $C^*$-algebras $\mathcal{A} = \mathcal{A}(\mathcal{H})$ associated (as in [1, 10.7.2]), with a continuous field of nonzero Hilbert spaces $\mathcal{H}$ on the locally compact Hausdorff space $T = \text{Prim } A$. (The existence of a suitable ideal follows from [1, 4.4.4, 4.5.4, 4.5.3, 10.5.8 and 10.7.7], referred to in that order.)

There exists a (unique) automorphism $f$ of $\mathcal{A}$ such that, for $x \in A$, $f(x) = \alpha(x)$. (First set $f' = (\alpha_i)_{i \in T}$, where, for each $i \in T$, $\alpha_i$ is the automorphism $x + t \mapsto \alpha(x) + t$ of $A/t$. Then $f'$ is an automorphism of $\mathcal{A}'$, the continuous field of $C^*$-algebras defined by $A$. By [1, 10.5.2] there is an isomorphism $\phi$ of $\mathcal{A}$ onto $\mathcal{A}'$ such that, for $x \in A$, $\phi(x) = (x + t)_{i \in T}$. Set $f = \phi^{-1} f'$.)

By [1, 10.7.9], there exists a nonempty open set $V \subset T$ such that $f|_V$ is defined by an automorphism $g$ of $\mathcal{H}|_V$.

Set $J_0$ equal to the set of continuous fields with respect to $\mathcal{A}$ which vanish outside $V$; $J_0$ is a nonzero closed two-sided ideal of $A$.

Choose some fixed bounded field of operators $h$ on $T$ such that $h|_V = g$. Then $hJ_0 \subset J_0$ and $J_0h \subset J_0$. (Suppose that $x \in J_0$. Since $x$ is continuous with respect to $\mathcal{A}$ and vanishes outside $V$, $hx$ and $xh$ are continuous and zero) at each point outside $V$. Also, since $g$ is an automorphism of $\mathcal{H}|_V$, $hx$ and $xh$ are continuous at each point in $V$ (see [1, 10.7.2]). Thus $hx$ and $xh$ are continuous at each point of $T$; therefore they are continuous with respect to $\mathcal{A}$.)

Thus $J_0$ is a closed two-sided ideal of the $C^*$-algebra of bounded fields of operators on $T$ generated by $J_0$ and $h$. Moreover, if $x \in J_0$, then for each $i \in V$ we have: $(\alpha(x))(t) = (f(x))(t) = f(t)(x(t)) = g(t)x(t)g(t)* = h(t)x(t)h(t)* = h(t)xh(t)*$, whence $\alpha(x) = hxh*$.

It follows by [1, 2.10.4] that $\alpha|_{J_0}$ is universally weakly inner.

3. Proof of Theorem 1. Let $\pi$ be a representation of $A$. Let $(E_i)_{i \in I}$ be a maximal family of mutually orthogonal nonzero central projections in the weak closure of $\pi(A)$ such that for each $i \in I$ there exists $U_i$ in the weak closure of $\pi(A)$ with $\pi(\alpha(x))E_i = U_i\pi(x)U_i^*E_i$ for all $x \in A$. If $\sum_{i \in E} E_i$ is not the largest projection in the weak closure of $\pi(A)$, let $\pi_0$ denote the (nonzero) representation of $A$ such that $\pi_0$ is the representation $x + \sum_{i \in E} E_i \mapsto \alpha(x) + \pi(\alpha(x))E_i$ of $J_0$. Applying Lemma 2 with $J = \text{Ker } \pi_0$ we get an ideal $J_0 \cap \text{Ker } \pi_0$ such that the automorphism $x + \text{Ker } \pi_0 \mapsto \alpha(x) + \text{Ker } \pi_0$ of $J_0/\text{Ker } \pi_0$ is universally weakly inner. Considering in particular the representation $x + \text{Ker } \pi_0 \mapsto \pi(\alpha(x))$ of $J_0/\text{Ker } \pi_0$, we get an element $U_0$ of the weak closure of $\pi_0(J_0)$ such that for all
$x \in J_0$, $\pi_0(\alpha(x)) = U_0\pi_0(x)U_0^*$. Letting $E_0$ denote the largest projection in the weak closure of $\pi_0(J_0)$, we have: $E_0 \neq 0$, $E_0$ is orthogonal to all $E_i$ ($i \in I$), both $E_0$ and $U_0$ are in the weak closure of $\pi(A)$, $E_0$ is central, and $\pi(\alpha(x))E_0 = U_0\pi(x)U_0^*E_0$ for all $x \in J_0$. By [1, 1.7.2], there exists an increasing approximate identity $(u_k)_{k \in K}$ for $J_0$; then $(\alpha(u_k))_{k \in K}$ is also such; by [1, 2.2.10], $E_0 = \sup_k \pi_0(u_k) = \sup_k \pi_0(\alpha(u_k))$; hence, for any $x \in A$, in the sense of strong convergence: $\pi(\alpha(x))E_0 = \lim_k \pi(\alpha(u_k))\pi(\alpha(x))E_0 = \lim_k \pi(\alpha(u_kx))E_0 = \lim_k U_0\pi(u_kx)U_0^*E_0 = \lim_k U_0\pi(u_k)\pi(x)U_0^*E_0 = U_0\pi(x)U_0^*E_0$. This contradicts the maximality of $(E_i)_{i \in I}$; $\sum_{i \in I} E_i$ must therefore be the largest projection in the weak closure of $\pi(A)$. Setting $U = \sum_{i \in I} E_i U_i$, we have: $U$ is in the weak closure of $\pi(A)$, and for all $x \in A$, $\pi(\alpha(x)) = U\pi(x)U^*$.

4. Problems. 4.1. If $\alpha$ is a universally weakly inner automorphism of a $C^*$-algebra must $\alpha(J) = J$ for every two-sided ideal $J$ (not necessarily closed)?

4.2. In [3], Lance raised the question, if $\alpha$ is an automorphism of a $C^*$-algebra such that $\alpha(J) \subseteq J$ for every closed two-sided ideal $J$, must $\alpha(J) = J$ for every closed two-sided ideal $J$? He showed that this is true for a postliminary $C^*$-algebra ([3, Theorem 1]).

4.3. The enveloping von Neumann algebra of a postliminary $C^*$-algebra is discrete. An automorphism of a discrete von Neumann algebra which fixes each element of the centre is inner. Can these facts be used to give a different proof of Theorem 1?

4.4. The condition in Theorem 1 that $A$ be postliminary is too strong, since, for example, Theorem 1 also holds for a discrete von Neumann algebra. However, Theorem 1 may fail to hold (e.g., [2, Example a]). Does Theorem 1 fail for every antiliminary $C^*$-algebra?

References

