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IDEAL PRESERVING AUTOMORPHISMS OF POSTLIMINARY C^*-ALGEBRAS

GEORGE A. ELLIOTT

ABSTRACT. E. C. Lance has shown that an automorphism of a separable postliminary C^*-algebra which preserves closed two-sided ideals is universally weakly inner. This note extends Lance's result to the nonseparable case.

1. Let A be a C^*-algebra, and let α be an automorphism of A. If π is a representation of A, α is *weakly inner* with respect to π if there exists an element U of the weak closure of $\pi(A)$ such that $\pi(\alpha(x)) = U\pi(x)U^*$ for all $x \in A$. Then necessarily UU^* is equal to U^*U and is the largest projection in the weak closure of $\pi(A)$; hence $\alpha(\text{Ker } \pi) = \text{Ker } \pi$. α is *universally weakly inner* if α is weakly inner with respect to every representation. This implies that $\alpha(J) = J$ for every closed two-sided ideal J of A.

THEOREM. *Suppose that A is a postliminary C^*-algebra* ([**1**, §4]). *Suppose that α is an automorphism of A such that $\alpha(J) = J$ for every closed two-sided ideal J of A. Then α is universally weakly inner.*

With the additional hypothesis that A be separable, this result is due to E. C. Lance ([**3**, Theorem 2]). In [**3**] Lance used the measure theoretic decomposition of a multiplicity-free representation of a separable C^*-algebra into irreducible representations ([**1**, §8]). The technique that we shall use here is the topological decomposition of a postliminary C^*-algebra into elementary C^*-algebras ([**1**, §10]). This in general is only possible locally; that is, after passing to a (nonzero) closed two-sided ideal. Accordingly, we shall first prove a local result (Lemma 2) and then deduce Theorem 1 with the aid of Zorn's lemma.

2. LEMMA. *Let A and α be as in Theorem* 1, *and let $J \neq A$ be a closed two-sided ideal of A. Then there exists a closed two-sided ideal $J_0 \supsetneq J$ such that the automorphism $x + J \mapsto \alpha(x) + J$ of J_0/J is universally weakly inner.*

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PROOF. Passing to the quotient of A by J, we may suppose that $J=0$.

Passing to a closed two-sided ideal of A, we may suppose that A is the C^*-algebra defined by the continuous field of elementary C^*-algebras $\mathcal{A}=\mathcal{A}(\mathcal{H})$ associated (as in [1, 10.7.2]), with a continuous field of nonzero Hilbert spaces $\mathcal{H}$ on the locally compact Hausdorff space $T=\text{Prim } A$. (The existence of a suitable ideal follows from [1, 4.4.4, 4.5.4, 4.5.3, 10.5.8 and 10.7.7], referred to in that order.)

There exists a (unique) automorphism f of $\mathcal{A}$ such that, for $x\in A$, $f(x)=\alpha(x)$. (First set $f'=(\alpha_t)_{t\in T}$, where, for each $t\in T$, α_t is the automorphism $x+t\mapsto\alpha(x)+t$ of A/t. Then f' is an automorphism of $\mathcal{A}'$, the continuous field of C^*-algebras defined by A. By [1, 10.5.2] there is an isomorphism φ of $\mathcal{A}$ onto $\mathcal{A}'$ such that, for $x\in A$, $\varphi(x)=(x+t)_{t\in T}$. Set $f=\varphi^{-1}f'\varphi$.)

By [1, 10.7.9], there exists a nonempty open set $V\subset T$ such that $f|V$ is defined by an automorphism g of $\mathcal{H}|V$.

Set J_0 equal to the set of continuous fields with respect to $\mathcal{A}$ which vanish outside V; J_0 is a nonzero closed two-sided ideal of A.

Choose some fixed bounded field of operators h on T such that $h|V=g$. Then $hJ_0\subset J_0$ and $J_0h\subset J_0$. (Suppose that $x\in J_0$. Since x is continuous with respect to $\mathcal{A}$ and vanishes outside V, hx and xh are continuous (and zero) at each point outside V. Also, since g is an automorphism of $\mathcal{H}|V$, hx and xh are continuous at each point in V (see [1, 10.7.2]). Thus hx and xh are continuous at each point of T; therefore they are continuous with respect to $\mathcal{A}$.)

Thus J_0 is a closed two-sided ideal of the C^*-algebra of bounded fields of operators on T generated by J_0 and h. Moreover, if $x\in J_0$, then for each $t\in V$ we have: $(\alpha(x))(t)=(f(x))(t)=f(t)(x(t))=g(t)x(t)g(t)^*=h(t)x(t)h(t)^*=hxh^*(t)$, whence $\alpha(x)=hxh^*$.

It follows by [1, 2.10.4] that $\alpha|J_0$ is universally weakly inner.

3. **Proof of Theorem 1.** Let π be a representation of A. Let $(E_i)_{i\in I}$ be a maximal family of mutually orthogonal nonzero central projections in the weak closure of $\pi(A)$ such that for each $i\in I$ there exists U_i in the weak closure of $\pi(A)$ with $\pi(\alpha(x))E_i=U_i\pi(x)U_i^*E_i$ for all $x\in A$. If $\sum_{i\in I}E_i$ is not the largest projection in the weak closure of $\pi(A)$, let π_0 denote the (nonzero) representation of $A: x\mapsto(1-\sum_{i\in I}E_i)(x)$. Applying Lemma 2 with $J=\text{Ker }\pi_0$ we get an ideal $J_0\supsetneq\text{Ker }\pi_0$ such that the automorphism $x+\text{Ker }\pi_0\mapsto\alpha(x)+\text{Ker }\pi_0$ of $J_0/\text{Ker }\pi_0$ is universally weakly inner. Considering in particular the representation $x+\text{Ker }\pi_0\mapsto\pi_0(x)$ of $J_0/\text{Ker }\pi_0$, we get an element U_0 of the weak closure of $\pi_0(J_0)$ such that for all

$x \in J_0$, $\pi_0(\alpha(x)) = U_0\pi_0(x)U_0^*$. Letting E_0 denote the largest projection in the weak closure of $\pi_0(J_0)$, we have: $E_0 \neq 0$, E_0 is orthogonal to all E_i ($i \in I$), both E_0 and U_0 are in the weak closure of $\pi(A)$, E_0 is central, and $\pi(\alpha(x))E_0 = U_0\pi(x)U_0^*E_0$ for all $x \in J_0$. By [1, 1.7.2], there exists an increasing approximate identity $(u_k)_{k\in K}$ for J_0; then $(\alpha(u_k))_{k\in K}$ is also such; by [1, 2.2.10], $E_0 = \sup_k \pi_0(u_k) = \sup_k \pi_0(\alpha(u_k))$; hence, for any $x \in A$, in the sense of strong convergence: $\pi(\alpha(x))E_0 = \lim_k \pi(\alpha(u_k))\pi(\alpha(x))E_0 = \lim_k \pi(\alpha(u_k x))E_0 = \lim_k U_0\pi(u_k x)U_0^*E_0 = \lim_k U_0\pi(u_k)\pi(x)U_0^*E_0 = U_0\pi(x)U_0^*E_0$. This contradicts the maximality of $(E_i)_{i\in I}$; $\sum_{i\in I} E_i$ must therefore be the largest projection in the weak closure of $\pi(A)$. Setting $U = \sum_{i\in I} E_i U_i$ we have: U is in the weak closure of $\pi(A)$, and for all $x \in A$, $\pi(\alpha(x)) = U\pi(x)U^*$.

4. **Problems.** 4.1. If α is a universally weakly inner automorphism of a C^*-algebra must $\alpha(J) = J$ for every two-sided ideal J (not necessarily closed)?

4.2. In [**3**], Lance raised the question, if α is an automorphism of a C^*-algebra such that $\alpha(J) \subset J$ for every closed two-sided ideal J, must $\alpha(J) = J$ for every closed two-sided ideal J? He showed that this is true for a postliminary C^*-algebra ([**3**, Theorem 1]).

4.3. The enveloping von Neumann algebra of a postliminary C^*-algebra is discrete. An automorphism of a discrete von Neumann algebra which fixes each element of the centre is inner. Can these facts be used to give a different proof of Theorem 1?

4.4. The condition in Theorem 1 that A be postliminary is too strong, since, for example, Theorem 1 also holds for a discrete von Neumann algebra. However, Theorem 1 may fail to hold (e.g. [**2**, Example a]). Does Theorem 1 fail for every antiliminary C^*-algebra?

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University of British Columbia, Vancouver, Canada