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REAL ZEROS OF A RANDOM SUM OF ORTHOGONAL POLYNOMIALS

MINAKETAN DAS

ABSTRACT. Let $c_0, c_1, c_2, \cdots$ be a sequence of normally distributed independent random variables with mathematical expectation zero and variance unity. Let $P_k^*(x)$ $(k=0, 1, 2, \cdots)$ be the normalised Legendre polynomials orthogonal with respect to the interval $(-1, 1)$. It is proved that the average number of the zeros of $c_0P_0^*(x)+c_1P_1^*(x)+\cdots+c_nP_n^*(x)$ in the same interval is asymptotically equal to $(3)^{-1/2}n$ when n is large.

1. Let $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x), \cdots$ be a sequence of polynomials orthogonal with respect to a given positive-valued weight function $\omega(x)$ over the interval (a, b) where one or both of a and b may be infinite and let $\psi_n(x)=g_n^{-1/2}\phi_n(x)$ with

$$g_n = \int_a^b \omega(x)\phi_n^2(x)dx.$$

Let $f(x)$ be defined by

$$f(x) \equiv f(\mathbf{c}; x) = \sum_{k=0}^{N} c_k\psi_k(x), \tag{1.1}$$

where the coefficients $c_0, c_1, c_2, \cdots$ form a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance unity. We take the ordered set $c_0, \cdots, c_n$ as the point $\mathbf{c}$ in an $(n+1)$-dimensional real vector space R_{n+1}. The probability that the point $\mathbf{c}$ lies in an "infinitesimal rectangle" $\Pi(\mathbf{c})$ with sides of lengths $dc_0, dc_1, \cdots, dc_n$ is

$$dP(\mathbf{c}) = \prod_{k=0}^{n} \left\{(2\pi)^{-1/2}\exp(-\tfrac{1}{2}c_k^2)dc_k\right\}.$$

Let $N(\mathbf{c}; \alpha, \beta)$ denote the number of zeros of the polynomial (1.1) in the interval $\alpha \leqq x \leqq \beta$. We establish the formula

$$\int_{R_{n+1}} N(\mathbf{c};\alpha,\beta)dP(\mathbf{c}) = \frac{1}{\pi}\int_\alpha^\beta \left[\frac{S_n(x)+R_n(x)}{D_n(x)} - \frac{1}{4}\frac{Q_n^2(x)}{D_n^2(x)}\right]^{1/2} dx, \tag{1.2}$$

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where

$$D_n(x) = \phi'_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'_n(x),$$
$$Q_n(x) = \phi''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi''_n(x),$$
$$R_n(x) = \tfrac{1}{2}\{\phi''_{n+1}(x)\phi'_n(x) - \phi'_{n+1}(x)\phi''_n(x)\}$$

and

$$S_n(x) = \tfrac{1}{6}\{\phi'''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'''_n(x)\}.$$

When n is large, we can find an estimate of the integrand in the right-hand side of (1.2) in terms of n and x only in an easily integrable form, since only two functions $\phi_n(x)$ and $\phi_{n+1}(x)$ are now involved.

Let $P_k^*(x)$ be the normalized Legendre polynomial $(k+\frac{1}{2})^{1/2}P_k(x)$, where

$$P_k(x) = \frac{1}{2^k}\frac{1}{k!}\frac{d^k}{dx^k}(x^2-1)^k;$$

the famous Legendre polynomial. Here $a=-1$ and $b=1$ and $\omega(x)\equiv 1$. Further $\psi_k(x)=P_k^*(x)$ with $g_n=(n+\frac{1}{2})^{1/2}$. We prove

THEOREM 1. *The average number of zeros of*

$$c_0P_0^*(x) + c_1P_1^*(x) + \cdots + c_kP_k^*(x) + \cdots + c_nP_n^*(x)$$

in $(-1, 1)$ *is asymptotically equal to* $n/\sqrt{3}$ *when* n *is sufficiently large.*

2. Let us put

$$A \equiv A_n(x) = \psi_0^2(x) + \psi_1^2(x) + \cdots + \psi_n^2(x),$$
$$B \equiv B_n(x) = \psi_1(x)\psi_1'(x) + \cdots + \psi_n(x)\psi_n'(x)$$

and

$$C \equiv C_n(x) = [\psi_1'(x)]^2 + \cdots + [\psi_n'(x)]^2.$$

Then, by Cauchy's inequality, $AC-B^2 \geqq [\psi_0\psi_1']^2 > 0$. By proceeding as in §3 of our earlier work (cf. [1]), we obtain

$$\text{(2.1)} \qquad \int_{R_{n+1}} N(\mathbf{c};\alpha,\beta)dP(\mathbf{c}) = \frac{1}{\pi}\int_\alpha^\beta \frac{(AC-B^2)^{1/2}}{A}dx.$$

We put $\lambda_n = h_n g_n^{-1} h_{n+1}^{-1}$, where h_n is the coefficient of x^n in $\phi_n(x)$ and g_n is defined as above. Then we have

$$\text{(2.2)} \qquad \sum_{k=0}^{n} g_k^{-1}\phi_k(x)\phi_k(y) = \lambda_n \frac{\phi_{n+1}(y)\phi_n(x) - \phi_{n+1}(x)\phi_n(y)}{y-x}.$$

This is the famous Christofel-Darboux formula [3, p. 135] in the theory of orthogonal functions. We set $y=x+\delta$ in the formula (2.2) and equate the coefficients of like powers of δ on both sides to obtain

$$\sum_{\nu=0}^{n} g_\nu^{-1}[\phi_\nu(x)]^2 = \lambda_n[\phi'_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'_n(x)], \tag{2.3}$$

$$\sum_{\nu=1}^{n} g_\nu^{-1}[\phi_\nu(x)\phi'_\nu(x)] = \frac{\lambda_n}{2}[\phi''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi''_n(x)] \tag{2.4}$$

and

$$\sum_{\nu=1}^{n} g_\nu^{-1}[\phi_\nu(x)\phi''_\nu(x)] = \frac{\lambda^n}{3}[\phi'''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'''_n(x)]. \tag{2.5}$$

Differentiating (2.4) and making use of (2.5), we get

$$\begin{aligned}\sum_{\nu=1}^{n} g_\nu^{-1}[\phi'_\nu(x)]^2 &= \frac{\lambda_n}{6}[\phi'''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'''_n(x)] \\ &\quad + \frac{\lambda_n}{2}[\phi''_{n+1}(x)\phi'_n(x) - \phi'_{n+1}(x)\phi''_n(x)].\end{aligned} \tag{2.6}$$

Making use of (2.3), (2.4) and (2.6), and the fact that $\lambda_n \neq 0$, we obtain the formula (1.2).

3. For Legendre polynomials $P_n(x)$, we have the relations

$$(1-x^2)P''_{n+1}(x) = 2xP'_{n+1}(x) - (n+1)(n+2)P_{n+1}(x) \tag{3.1}$$

and

$$(1-x^2)P''_n(x) = 2xP'_n(x) - n(n+1)P_n(x). \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\begin{aligned}&(1-x^2)[P''_{n+1}(x)P'_n(x) - P''_n(x)P'_{n+1}(x)] \\ &\quad = -(n+1)[n\{P_{n+1}(x)P'_n(x) - P_n(x)P'_{n+1}(x)\} + 2P_{n+1}(x)P'_n(x)]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&(1-x^2)[P''_{n+1}(x)P_n(x) - P_{n+1}(x)P''_n(x)] \\ &\quad = 2x[P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)] - 2(n+1)P_n(x)P_{n+1}(x).\end{aligned} \tag{3.4}$$

Differentiating (3.4) and using (3.3), we get

$$(3.5)\quad \begin{aligned}(1-x^2)&[P'''_{n+1}(x)P_n(x) - P'''_n(x)P_{n+1}(x)]\\ &= (n+1)[n\{P_{n+1}(x)P'_n(x) - P_n(x)P'_{n+1}(x)\} + 2P_{n+1}(x)P'_n(x)]\\ &\quad + \frac{16}{1-x^2}[x\{P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)\}\\ &\qquad\qquad - (n+1)P_n(x)P_{n+1}(x)]\\ &\quad + 2(n+1)[P'_n(x)P_{n+1}(x) - P_n(x)P'_{n+1}(x)].\end{aligned}$$

We recall another formula for the derivative of a Legendre function [**2**, p. 179, (17)], viz.:

$$(3.6)\qquad (x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$$

and

$$(3.7)\qquad (x^2-1)P'_{n+1}(x) = (n+1)xP_{n+1}(x) - (n+1)P_n(x).$$

The application of (3.6) and (3.7) yields

$$(3.8)\quad \begin{aligned}(x^2-1)&[P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)]\\ &= (n+1)[2xP_n(x)P_{n+1}(x) - P_n^2(x) - P_{n+1}^2(x)],\end{aligned}$$

$$(3.9)\quad \begin{aligned}(x^2-1)&[P'_{n+1}(x)P_n(x) + P_{n+1}(x)P'_n(x)]\\ &= (n+1)[P_{n+1}^2(x) - P_n^2(x)]\end{aligned}$$

and

$$(3.10)\quad (x^2-1)P_{n+1}(x)P'_n(x) = (n+1)P_{n+1}(x)[P_{n+1}(x) - xP_n(x)].$$

To evaluate

$$P_n^2(x)+P_{n+1}^2(x)-2xP_n(x)P_{n+1}(x),$$

we set $x=\cos\gamma$ and make use of the celebrated Laplace's formula (cf. [**2**, p. 208]) giving the asymptotic value of $P_n(\cos\gamma)$ as

$$\left(\frac{2}{\pi n\sin\gamma}\right)^{1/2}\cos\left[\left(n+\frac{1}{2}\right)\gamma - \frac{\pi}{4}\right] + O((n\sin\gamma)^{-3/2})$$

in the range $\epsilon<\gamma<\pi-\epsilon$, where $0<\epsilon<\pi/2$. After some simplifications, we find

$$P_n^2(x) + P_{n+1}^2(x) - 2xP_n(x)P_{n+1}(x)$$
$$= \frac{2}{\pi n \sin\gamma}\left\{\cos^2\left[\left(n+\frac{1}{2}\right)\gamma - \frac{\pi}{4}\right] + \cos^2\left[\left(n+\frac{3}{2}\right)\gamma - \frac{\pi}{4}\right]\right.$$
$$\left. - 2\cos\gamma\cos\left[\left(n+\frac{1}{2}\right)\gamma - \frac{\pi}{4}\right]\cos\left[\left(n+\frac{3}{2}\right)\gamma - \frac{\pi}{4}\right]\right\}$$
$$+ O(n^{-2}\operatorname{cosec}^2\gamma)$$
$$= \frac{2}{\pi n}(1-x^2)^{1/2} + O(n^{-2}(1-x^2)^{-1}).$$

Making use of (3.8), we obtain

$$\{P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)\} > \frac{2}{\pi}(1-x^2)^{-1/2} \tag{3.11}$$

for sufficiently large n and $|x| < 1 - n^{-2/3}\log n$. By the first theorem of Stieltjes, [2, p. 197,(8)] $|P_n(x)| \leqq 4n^{-1/2}(1-x^2)^{-1/4}$ and by (3.6), $|P'_n(x)| \leqq 8n^{1/2}(1-x^2)^{-5/4}$. Thus

$$nP_n(x)P_{n+1}(x) = O((1-x^2)^{-1/2}), \tag{3.12}$$

$$P_n(x)P'_n(x) = O((1-x^2)^{-3/2}) \tag{3.13}$$

and

$$P'_{n+1}(x)P_n(x) + P'_n(x)P_{n+1}(x) = O((1-x^2)^{-3/2}). \tag{3.14}$$

By putting these estimates in (3.3), (3.4) and (3.5), we get

$$(1-x^2)(P''_{n+1}P'_n - P''_nP'_{n+1})$$
$$= n(n+1)(P'_{n+1}P_n - P'_nP_{n+1}) + O(n(1-x^2)^{-3/2}),$$
$$(1-x^2)(P''_{n+1}P_n - P''_nP_{n+1})$$
$$= 2x(P'_{n+1}P_n - P'_nP_{n+1}) + O((1-x^2)^{-3/2})$$

and

$$(1-x^2)(P'''_{n+1}P_n - P'''_nP_{n+1}) = \left\{\frac{16x}{1-x^2} - n - n^2\right\}(P'_{n+1}P_n - P'_nP_{n+1})$$
$$+ O(n(1-x^2)^{-3/2}),$$

where we have written P_k, P_k', P_k'' and P_k''' for $P_k(x)$, $P_k'(x)$, $P_k''(x)$ and $P_k'''(x)$, respectively. This abbreviation is also employed below.

By using (3.11), we finally obtain, for $|x| < 1 - n^{-2/3} \log n$, the estimate

$$(P_{n+1}'''P_n - P_{n+1}P_n''')/(P_{n+1}'P_n - P_{n+1}P_n') = -n^2(1-x^2)^{-1}(1+O(1/n)),$$

$$(P_{n+1}''P_n' - P_{n+1}'P_n'')/(P_{n+1}'P_n - P_{n+1}P_n') = n^2(1-x^2)^{-1}(1+O(1/n))$$

and

$$(P_{n+1}''P_n - P_{n+1}P_n'')/(P_{n+1}'P_n - P_{n+1}P_n') = O(n^{2/3}(1-x^2)^{-1}).$$

Putting these values for $Q_n(x)/D_n(x)$, $R_n(x)/D_n(x)$ and $S_n(x)/D_n(x)$ in (1.2), the expression enclosed by brackets is estimated by

$$\frac{n^2}{3}\frac{1}{(1-x^2)}\left[1 + O\left(\frac{1}{(\log n)^3}\right)\right]. \tag{3.15}$$

Let $\epsilon = n^{-2/3} \log n$ and $N(\mathbf{c}; \epsilon)$ denote the number of zeros of

$$f(\mathbf{c}; x) = c_0 \overset{*}{P}_0(x) + c_1 \overset{*}{P}_1(x) + \cdots + c_n \overset{*}{P}_n(x)$$

in $-1+\epsilon \leqq x \leqq 1-\epsilon$. By (1.2), we have

$$\int_{R_{n+1}} N(\mathbf{c}; \epsilon) dP(\mathbf{c}) = \frac{1}{\pi}\int_{-1+\epsilon}^{1-\epsilon} \frac{n}{\sqrt{3}}[1 + O((\log n)^{-3})](1-x^2)^{-1/2}dx$$
$$= \frac{n}{\sqrt{3}}\{1 + O((\log n)^{-3})\}.$$

To complete the proof of Theorem 1, we observe (cf. [2, p. 250]) that

$$|P_n(z)| = \frac{1}{\pi}\left|\int_0^{\pi} \{z + i(1-z^2)^{1/2}\cos \tau\}^n d\tau\right|$$

and thus for $z = 1 + \epsilon e^{i\theta}$, $|P_n(z)| < (1+3\epsilon)^n < 2n^3 \exp(n^{1/3})$. Further $P_n(1) = 1$. We can prove, as in our work [1, p. 722], that

$$\Pr\left(\max_{0 \leqq k \leqq n} |c_k| \leqq n\right) > 1 - e^{-n/3},$$

so that for $z = 1 + \epsilon e^{i\theta}$, we have

$$(\Pr(|f(\mathbf{c}; z)| \geqq 4n^4 \exp(n^{1/3}))) < \exp(-n^2/3)$$

and

$$(\Pr(|f(\mathbf{c}; z)| < 1)) < 1/n.$$

Let $\nu(\mathbf{c}; \epsilon)$ denote the number of zeros of $|f(\mathbf{c}; z)|$ in $|z-1| \leqq \epsilon$. By making use of Jensen's theorem, we find

$$\nu(\mathbf{c}; \epsilon) \log 2 \leqq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(\mathbf{c}; 1 + \epsilon e^{i\theta})}{f(\mathbf{c}; 1)} \right| d\theta = O(n^{1/3})$$

with probability at least equal with $1-2/n$. This shows that the average number of zeros of $f(\cdot\,; x)$ in $1-\epsilon \leqq x \leqq 1$ and similarly in $-1 \leqq x \leqq -1+\epsilon$ is $O(n^{1/3})$. Therefore, on using (3.16), we finally obtain the proof of Theorem 1.

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Fakir Mohan College, Balasore, Orissa, India