ON THE UNIVALENCE OF A CERTAIN INTEGRAL

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Abstract. We consider the function \( g(z) = \int_0^{t} \left[ \frac{f(t)}{t} \right]^{\alpha} dt \) for \( f \) in the classes of convex, starlike, and close-to-convex univalent functions, and we determine precisely which values of \( \alpha \) yield a close-to-convex function \( g \).

1. Introduction. Let \( S \) denote the class of functions \( f(z) = z + a_2 z^2 + \cdots \) that are analytic and univalent in the unit disk \( D = \{ z : |z| < 1 \} \). Let \( C, S^* \), and \( K \) denote respectively the subclasses of \( S \) whose members are close-to-convex \([2]\), starlike relative to the origin, and convex in \( D \). For \( f \in S \), set

\[
g(z) = \int_0^{t} \left[ \frac{f(t)}{t} \right]^{\alpha} dt,
\]

where \( \alpha \) is real. Causey \([1]\) has proved that if \( f \) is close-to-convex relative to a function \( \phi \in K \), and \( 0 \leq \alpha \leq 1 \), then \( g \) is close-to-convex relative to a function in \( K \). Nunokowa \([4]\) recently showed that \( g \in S \) provided \( f \in S^* \), \( 0 \leq \alpha \leq 3/2 \), or \( f \in K \), \( 0 \leq \alpha \leq 3 \). In this paper, the following sharp theorems are proved.

**Theorem 1.** If \( f \in S^* \), then the function \( g \) is in \( C \) provided \(-1/2 \leq \alpha \leq 3/2 \). If \( \alpha \in [ -1/2, 3/2 ] \), there is a function \( f \in S^* \) such that the corresponding \( g \) is not in \( S \).

**Theorem 2.** If \( f \in K \), then \( g \) is in \( C \) provided \(-1 \leq \alpha \leq 3 \). For \( \alpha \in [ -1, 3 ] \), there is a function \( f \in K \) such that the corresponding function \( g \) is not in \( S \).

**Theorem 3.** If \( f \in C \), then \( g \) is in \( C \) provided \(-1/2 \leq \alpha \leq 1 \). If \( \alpha \in [ -1/2, 1 ] \), there is a function \( f \in C \) such that the corresponding \( g \) is not in \( C \).

2. Proofs of Theorems 1 and 2. Kaplan \([2]\) has shown that \( g \in C \) if and only if

\[
\int_{\theta_1}^{\theta_2} \text{Re} \left[ 1 + r e^{i\phi} \frac{g''(re^{i\phi})}{g'(re^{i\phi})} \right] d\phi > -\pi
\]

whenever \( 0 \leq r < 1 \), \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \). For \( f \in S^* \), we have \( \text{Re} \{ zf'/f \} > 0 \) for \( z \in D \). By (1) it follows that for \( \alpha > 0 \)

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The last quantity is not less than \(-\pi\) provided \(0 \leq \alpha \leq 3/2\), and hence \(g \in C\). In fact, for \(0 \leq \alpha \leq 1\),

\[
\text{Re} \left[ 1 + \frac{2g''(z)}{g'(z)} \right] = \alpha \frac{zf'(z)}{f(z)} + 1 - \alpha \geq 1 - \alpha \geq 0,
\]

which implies that \(g \in K \subseteq C\). Finally, when \(-1/2 \leq \alpha \leq 0\), a result of Marx [3] gives \(\text{Re} \left[ (z)/z \right] > 0\). Hence, \(\text{Re} g'(z) = \text{Re} \left[ (z)/z \right] > 0\), and this implies \(g \in C\).

For \(\alpha \in [-1/2, 3/2]\), set \(f(z) = z/(1-z)^2\). Then \(g'(z) = (1-z)^{-2\alpha}\), and thus \(g(z)\) is univalent in \(E\) if and only if \(h(z) = (1-z)^{1-2\alpha}\) is univalent in \(E\). By a lemma due to Royster [5], the latter is the case if and only if \(-1/2 \leq \alpha \leq 3/2\), \(\alpha \neq 1/2\). When \(\alpha = 1/2\), \(g(z) = -\ln(1-z)\) which is univalent in \(E\). This completes the proof of Theorem 1.

The proof of Theorem 2 parallels that of the first theorem. If \(f \in K\), we utilize the facts that \(\text{Re} \left[ (z)/z \right] \geq 1/2\) and \(\text{Re} \left[ z/f \right] > 0\) for \(z \in E\) [3]. Sharpness follows by consideration of \(f(z) = z/(1-z)\).

3. Proof of Theorem 3. The following lemma generalizes a result of Sakaguchi [6].

**Lemma.** Let \(f(z) = \sum_{n=1}^{\infty} a_n z^n\), \(g(z) = \sum_{n=1}^{\infty} b_n z^n\) be analytic in \(E\) and let \(g(z)\) be univalent and starlike relative to the origin in \(E\). If \(H\) denotes the convex hull of the image of \(E\) under the mapping \(f'/g'\), then \(f(z)/g(z) \in H\) for \(z \in E\).

**Proof.** Let \(\psi(w)\) denote the inverse function of \(g(z)\) and let \(h(w) = f(\psi(w))\). Then

\[
\frac{f(z)}{g(z)} = \frac{h(w)}{w} = \frac{1}{w} \int_0^w h'(t) dt = \frac{1}{w} \int_0^w \frac{f'(\psi(t))}{g'(\psi(t))} dt = \frac{1}{\rho} \int_0^\rho \frac{f'(\rho e^{i\theta})}{g'(\rho e^{i\theta})} d\rho,
\]

where \(w = re^{i\phi}\), and the result follows.

Suppose \(f \in C\). Then there exists a convex function \(\phi(z)\), \(\phi(0) = 0\), in \(E\) such that \(\text{Re} \left[ f'/\phi' \right] > 0\) for \(z \in E\) [2]. If \(0 \leq \alpha \leq 1\), set

\[
\Phi(z) = \int_0^z \left[ \frac{\phi(t)}{t} \right] dt.
\]
Then as in (3) $\Phi$ is convex.

Now,

$$\text{Re} \left[ \frac{g'(z)}{\Phi'(z)} \right] = \text{Re} \left[ \frac{f(z)}{\phi(z)} \right]^\alpha > 0$$

by (1) and the lemma. This proves $g \in C$ in this case.

When $-1/2 \leq \alpha < 0$, set

$$\Phi(z) = e^{-i(1+\alpha)\beta} \int_0^z \frac{\phi(t)}{t} \frac{1}{1+2a} \, dt$$

where $\beta = \arg \phi'(0)$. As in the previous case, it follows that $\Phi(z)$ is convex in $E$. Now

$$\text{Re} \left[ \frac{g'(z)}{\Phi'(z)} \right] = \text{Re} \left\{ \left[ \frac{f(z)}{\phi(z)} \right]^\alpha \left[ e^{i\beta} \frac{z}{\phi(z)} \right]^{1+\alpha} \right\}.$$ 

Since $\text{Re} \left\{ z/h(z) \right\} > 0$ when $h(z) \in K [3]$, we conclude by the lemma that (4) is positive and, hence, that $g \in C$ for $-1/2 \leq \alpha < 0$.

When $\alpha < -1/2$, the sharpness is a consequence of Theorem 1. Suppose $\alpha > 1$. The function

$$f(z) = \frac{z(1+\mu z)}{(1+z)^2}, \quad \mu = (\cos \gamma)e^{i\gamma}, \quad 0 < \gamma < \pi,$$

is close-to-convex with respect to $\phi(z) = -ie^{i\gamma}z/(1+\gamma)$ and maps $E$ onto the plane minus the slit

$$w = (1 + i \cot \gamma)/4 + i[1/2 - (1 + i \cot \gamma)/4], \quad 0 \leq t < \infty,$$

with $f(e^{i(\pi-2\gamma)}) = (1+i \cot \gamma)/4$. If $\theta_1 = \pi - 2\gamma$ and $\theta_1 < \theta_2 < \pi$, then as $r \to 1$ and $\theta_2 \to \pi$, we have

$$\arg f(re^{i\theta}) - \arg f(re^{i\theta_1}) \to \gamma - \pi/2 - \arctan(\cot \gamma).$$

This in turn approaches $-\pi$ as $\gamma \to 0^+$. Thus, for $\alpha > 1$, we have

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ 1 + re^{i\theta} \frac{g''(re^{i\theta})}{g'(re^{i\theta})} \right\} d\theta$$

$$= (1 - \alpha)(\theta_2 - \theta_1) + \alpha \int_{\theta_1}^{\theta_2} \text{Re} \left\{ re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right\} d\theta$$

$$< \alpha \{ \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \}.$$ 

The last quantity can be made arbitrarily close to $-\alpha \pi$ by choosing $r, \theta, \gamma$ near $1, \pi, 0$ respectively. By (2) it follows that $g \in C$. 

4. A related problem. Let \( f \in S \) and set \( G(z) = \int_0^z [f'(t)]^\alpha dt \). If \( f \in K \), then \( zf'(z) \) is in \( S^* \). Hence, by Theorem 1, \( G \in C \) whenever \( f \in K \) if and only if \(-1/2 \leq \alpha \leq 3/2\). The following result extends a theorem due to Royster [5].

**Theorem 4.** If \( f \in C \), then \( G \in C \) provided \(-1/3 \leq \alpha \leq 1\). If \( \alpha \in [-1/3, 1] \), there is a function \( f \in C \) such that the corresponding \( G \in S \).

**Proof.** The result in case \( 0 \leq \alpha \leq 1 \) was proved by Royster [5]. Suppose \(-1/3 \leq \alpha < 0\). If \( f \in C \), there is a convex function \( \phi \) in \( E \) such that \( \phi(0) = 0 \) and \( \text{Re} \{ f'(z)/\phi'(z) \} > 0 \) for \( z \in E \). Set

\[
\Phi(z) = e^{-i\theta(1+2\alpha)} \int_0^z \left[ \frac{\phi(t)}{t} \right]^{1+2\alpha} dt,
\]

where \( \beta = \text{arg } \phi'(0) \). Then, as in (3), \( \Phi(z) \) is convex in \( E \) and

\[
\frac{G'(z)}{\Phi'(z)} = \left[ \frac{z\phi'(z)}{\phi(z)} \right]^\alpha \left[ e^{i\theta} \frac{z}{\phi(z)} \right]^{1+2\alpha}.
\]

Since \( \text{Re} \{ z\phi'/\phi \} > 0 \), \( \text{Re} \{ e^{i\theta}z/\phi(z) \} > 0 \), it follows that \( \text{Re} \{ G'(z)/\phi'(z) \} > 0 \) for \( z \in E \) and, hence, that \( G \in C \).

In order to prove the result is sharp, let \( f(z) = z(1-z)/(1-z)^2 \), which is close-to-convex relative to \( z/(1-z) \). Then, \( G(z) = -\log(1-z) \) if \( \alpha = 1/3 \) and

\[
G(z) = \frac{1}{1-3\alpha} \left[ (1-z)^{1-3\alpha} - 1 \right]
\]

if \( \alpha \neq 1/3 \). By a lemma of Royster [5], \( G \in S \) if \( \alpha \in [-1/3, 1] \).

**References**


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