L² ASYMPTOTES FOR THE KLEIN-GORDON EQUATION

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ABSTRACT. An approximation a(x, t) is obtained for solutions u(x, t) of the Klein-Gordon equation. a(x, t) can be expressed in terms of the Fourier transforms of the Cauchy data and it is shown that \( \|a(\cdot, t) - u(\cdot, t)\| \to 0 \) as \( t \to \infty \). This result is applied to show how energy distributes among various conical regions.

A wide class of solutions to the Klein-Gordon equation

\[
\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial^2 u}{\partial t^2} = u
\]

can be written in the form

\[
u(x, t) = (2\pi)^{-n/2} \int e^{ix \cdot y} \left[ F(y) \cos t \sqrt{1 + y^2} + G(y) \sin t \sqrt{1 + y^2} \right] d^n y
\]

\[
= (2\pi)^{-n/2} \int e^{ix \cdot y} \left[ \psi(y) \exp \left[ it \sqrt{1 + y^2} \right] + \phi(y) \exp \left[ -it \sqrt{1 + y^2} \right] \right] d^n y
\]

where \( F = \psi + \phi, \ G = i(\psi - \phi) \) are in \( L^2(\mathbb{R}^n) \) and the integral over \( \mathbb{R}^n \) is interpreted in the sense of Plancherel's theorem. Our main result is

**Theorem 1.** Define \( a(x, t) = 0 \) for \( |x| > t \) and for \( |x| < t \) define

\[
a(x, t) = \left\{ e^{\theta(x, t)} \psi \left( \frac{-x}{\sqrt{t^2 - x^2}} \right) + e^{-\theta(x, t)} \phi \left( \frac{x}{\sqrt{t^2 - x^2}} \right) \right\} \rho(x, t),
\]

\[
\theta(x, t) \equiv n\pi/4 + \sqrt{(t^2 - x^2)}, \quad \rho(x, t) \equiv t(t^2 - x^2)^{(n+1)/4}
\]

where \( \psi \) and \( \phi \) are the same \( L^2 \) functions as in (1). Then

\[
\| u(\cdot, t) - a(\cdot, t) \|_2 \equiv \int | u(x, t) - a(x, t) |^2 d^n x \to 0
\]
as \( t \to \infty \).
Before starting on the proof of Theorem 1 we mention a corollary. Define

\[ \chi(x, t) = \begin{cases} 1 & \text{if } |x| < t, \\ 0 & \text{otherwise}. \end{cases} \]

**Corollary 1.**

\[ \lim_{t \to \infty} \|\chi u(\cdot, t)\|_2^2 = \|\psi\|_2^2 + \|\phi\|_2^2 = (\|F\|_2^2 + \|G\|_2^2)/2. \]

**Proof.** By Theorem 1, \( \lim_{t \to \infty}\|(1 - \chi)u(\cdot, t)\|_2 = 0 \) and hence

\[ \lim_{t \to \infty} \|\chi u(\cdot, t)\|_2^2 = \|\chi u(\cdot, t)\|_{L^2}^2 = 0. \]

Thus the corollary follows from the fact (see Brodsky [1]) that

\[ \lim_{t \to \infty} ||u(\cdot, t)||_2^2 = (\|F\|_2^2 + \|G\|_2^2)/2. \]

**Remark.** Let \( V_n \) denote the volume of the unit ball in \( \mathbb{R}^n \) so that

\[ \|\chi u(\cdot, t)\|_2^2 \leq V_n t^n \|\chi u(\cdot, t)\|_2^2. \]

Applying the corollary one sees that if \( \|\chi u(\cdot, t)\|_2 \to 0 \) as \( t \to \infty \) then \( u = 0 \), a special case of a result by Littman [3].

The above corollary can be extended to the case where \( \chi \) is replaced by the characteristic function of other cones (see Corollary 1'). Theorem 1 will be deduced from Theorem 1' below. The proof of Theorem 1' is based on

**Lemma 1.** Define

\[ W_t(x) = (2\pi)^{-n/2} \int e^{ix \cdot v} \exp\left[-i\sqrt{1 + v^2}\right](1 + v^2)^{-(n+2)/4} dv dy. \]

**Then, for every** \( f \in L^1(\mathbb{R}^n) \) and \( \lambda \in \mathbb{R}^n \),

\[ \lim_{t \to \infty} t^{n/2} e^{i\pi x^2} \exp[\pi t/(1 + x^2)] W_t \ast f \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right) = \hat{f}(\lambda). \]

**Proof.** In [5] it is shown that for \( t > 0 \),

\[ W_t(x) = t^{-n/2} e^{-i\pi x} \exp[\pi t/(1 + x^2)] + R_t(x) \]

where \( \|R_t\|_\infty = O(t^{-1-n/2}) \) as \( t \to \infty \). By \( \sqrt{(x^2 - t^2)} \) we mean the value that lies on the positive imaginary axis when \( |x| < t \) and on the positive real axis when \( |x| > t \). Since \( \|R_t\|_\infty = O(t^{-1-n/2}) \) it is clear that \( t^{n/2} \|R_t \ast f\|_\infty = O(t^{-1}) \) and hence
\[
\lim_{t \to \infty} t^{n/2} e^{i \omega \sqrt{t}} R_t f \left( \frac{\lambda t}{\sqrt{1 + \lambda^2}} \right) = 0.
\]
Thus to complete the proof of the lemma we must show
\[
\lim_{t \to \infty} \left( 2\pi \right)^{-n/2} \int \exp \left( \frac{it}{\sqrt{1 + \lambda^2}} \right)
- \sqrt{\left( \frac{\lambda t}{\sqrt{1 + \lambda^2}} - x \right)^2 - t^2} \right) f(x) \, d^n x
= \left( 2\pi \right)^{-n/2} \int e^{-\lambda \cdot x} f(x) \, d^n x \equiv \hat{f}(\lambda).
\]
But this is a consequence of Lebesgue's dominated convergence theorem because
\[
\lim_{t \to \infty} \left( 2\pi \right)^{-n/2} \int \exp \left( \frac{it}{\sqrt{1 + \lambda^2}} \right)
- \sqrt{\left( \frac{\lambda t}{\sqrt{1 + \lambda^2}} - x \right)^2 - t^2} \right) f(x) \, d^n x
= \left( 2\pi \right)^{-n/2} \int e^{-\lambda \cdot x} f(x) \, d^n x \equiv \hat{f}(\lambda).
\]
For \( \phi \in L^2(\mathbb{R}^n) \) define
\[
(3) \quad U_t \phi(x) = \left( 2\pi \right)^{-n/2} \int e^{ix \cdot y} \exp \left( -it \sqrt{1 + y^2} \right) \phi(y) \, d^n y.
\]
**Corollary 2.** If there exists \( f \in L_1(\mathbb{R}^n) \) such that \( \hat{f}(\lambda) = (1 + \lambda^2)^{(n+2)/4} \cdot \phi(\lambda) \) then

(i) \( \| U_t f \|_\infty \leq (2\pi)^{-n/2} \| f \|_1 \| W_t \|_\infty \sim (2\pi)^{-n/2} \| f \|_1 t^{-n/2} \) as \( t \to \infty \),

(ii) \( \lim_{t \to \infty} t^{n/2} e^{i \alpha(t, \lambda)} U_t \phi \left( \frac{\lambda t}{\sqrt{1 + \lambda^2}} \right) = \hat{f}(\lambda) = (1 + \lambda^2)^{(n+2)/4} \phi(\lambda) \),

where \( \alpha(\lambda, t) \equiv n\pi/4 + t/\sqrt{1 + \lambda^2} \).

**Proof.** Both (i) and (ii) follow easily from the fact that
\[
U_t \phi(x) = \left( 2\pi \right)^{-n/2} \int e^{ix \cdot y} \exp \left( -it \sqrt{1 + y^2} \right) (1 + \lambda^2)^{-(n+2)/4} \hat{f}(\lambda) \, d^n y
= \left( 2\pi \right)^{-n/2} \int W_t(x - y) \hat{f}(y) \, d^n y \equiv W_t * f(x).
\]

**Remark.** The proof of Corollary 2 follows the approach used by Brodsky [2] and Segal [7, pp. 95–98] to obtain bounds like that given by (i). Recently, I became aware of a different approach by Littman [4] which when applied to the present situation yields (i) and (ii) with different assumptions on \( \phi \). Using Littman's approach Theorem 1 can be extended to more general situations (see [6]).
Motivated by (ii) of Corollary 2 we define an approximation $A \phi(x)$ to $U \phi(x)$ by requiring

$$t^{n/2}e^{ia(x,t)} A \phi \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right) = (1 + \lambda^2)^{(n+2)/4} \phi(\lambda), \quad \lambda \in \mathbb{R}^n, \ t > 0.$$

To see how this works out consider the transformation

$$T_t: \lambda \rightarrow x = \frac{\lambda t}{\sqrt{(1 + \lambda^2)}}$$

that maps $\mathbb{R}^n$ onto the ball $|x| < t$. Since $x^2 = \lambda^2 t^2/(1 + \lambda^2)$ we have $\lambda^2(t^2 - x^2) = x^2$ and hence $T_t^{-1}: x \rightarrow \lambda = x/\sqrt{(t^2 - x^2)}$. Taking $\lambda = T_t^{-1}(x)$ in (4) and multiplying by $t^{-n/2}e^{-ia}$ gives

$$A_t \phi(x) = \exp[-ia(T_t^{-1}(x), t)] t^{-n/2} \left( 1 + \frac{x^2}{t^2 - x^2} \right)^{(n+2)/4} \phi(T_t^{-1}(x))$$

(5)

$$= e^{-\omega(x,t)} \rho(x,t) \phi \left( \frac{x}{\sqrt{(t^2 - x^2)}} \right), \quad |x| < t,$$

where $\omega$ and $\rho$ are the functions defined in Theorem 1.

**Theorem 1'.** For $\phi \in L^2(\mathbb{R}^n)$ define $U \phi$ and $A \phi$ by (3) and (5).

Then

(i) $\|A \phi\|_{2,t} = \|\phi\|_2 = \|U \phi\|_2$,

(ii) $\lim_{t \to \infty} \|U \phi - A \phi\|_{2,t} = 0$,

(iii) $\lim_{t \to \infty} \int_{|x| > t} |U \phi(x)|^2 dx = 0$,

where

$$\|f\|_{2,t} = \left\{ \int_{|x| < t} |f(x)|^2 d^nx \right\}^{1/2}.$$

**Proof.** It is not difficult to check that the Jacobian of $T_t$ is

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(\lambda_1, \ldots, \lambda_n)} = t^n(1 + \lambda^2)^{-(n+2)/2} = \rho^{-2}(T_t(\lambda), t).$$

Thus by the change of variable theorem for multiple integrals

$$\|f\|_{2,t}^2 = \int |f(T_t(\lambda))|^2 \rho^{-2}(T_t(\lambda), t) d^nx\lambda.$$

Taking $f = A \phi$ in (6) we obtain the left-hand side of (i). The other half of (i) is Parseval's equality.

To prove (ii) let $\epsilon > 0$ and choose $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\|\phi - \phi\|_2 < \epsilon/3$. Applying (i) to $\phi - \phi$ we have
Thus
\[ \| U \phi - A \phi \|_{2,t} \leq \| U \phi - U \phi \|_{2,t} + \| U \phi - A \phi \|_{2,t} = \| A \phi - A \phi \|_{2,t} = \| \phi - \phi \|_2. \]

so to prove (ii) we need only show there exists \( \tau \) such that \( t > \tau \) implies \( \| U \phi - A \phi \|_2 < \epsilon/3 \). Since this amounts to proving (ii) with \( \phi \) replaced by \( \phi \), we simply assume \( \phi \in C_c^\infty (R^n) \).

Applying (6) we have
\[ \| U \phi - A \phi \|_{2,t}^2 = \int | U \phi \left( \frac{\lambda t}{\sqrt{1 + x^2}} \right) - A \phi \left( \frac{\lambda t}{\sqrt{1 + x^2}} \right) |^2 \cdot e^{-t^2(T_n(\lambda), t, d^n \lambda)} \]
\[ = \int \left| e^{-i\phi(x,t)} [ g_t(\lambda) - \phi(\lambda) ] \right|^2 d^n \lambda \]
where
\[ g_t(\lambda) = e^{i\phi(x,t)} e^{n+1/2(1 + \lambda^2)} \int U \phi \left( \frac{\lambda t}{\sqrt{1 + x^2}} \right) d^n \lambda. \]

Clearly
\[ \| U \phi \|_{2,t}^2 = \int \| g_t(\lambda) \|^2 d^n \lambda \]
and hence
\[ \| g_t \|_{2,t} \leq \| U \phi \|_2 = \| \phi \|_2. \]

Since \( \phi \in C_c^\infty (R^n) \) part (ii) of Corollary 2 can be used to conclude that for every \( \lambda \in R^n \)
\[ \lim_{t \to \infty} g_t(\lambda) = \phi(\lambda). \]

An application of Fatou's lemma and Egoroff's theorem shows that
\[ (8) \lim_{t \to \infty} \| g_t - \phi \|_2 = 0 \] which in view of (7) establishes (ii).

To prove (iii) we must show \( \lim_{t \to 0} \| U \phi \|_{2,t}^2 - \| U \phi \|_2^2, t = 0 \) which is obvious from (i) and (ii).

**Proof of Theorem 1.** Since
\[ u(x, t) = \overline{U \phi(-x)} + U \phi(x), \quad a(x, t) = \overline{A \phi(-x)} + A \phi(x), \]
Theorem 1 follows immediately from Theorem 1'.

Let \( \Gamma \) be a cone inside the cone \( \{ (x, t) : |x| < t \} \). That is, assume \( (x, t) \in \Gamma \) implies \( |x| < t \) and \( (sx, st) \in \Gamma \) for all \( s > 0 \). Put
and let $\beta(\lambda)$ be the characteristic function of $B$. Then the characteristic function $\gamma(x, t)$ of $\Gamma$ is zero for $|x| > t$ and satisfies

$$\gamma(x, t) = \beta \circ T_t^{-1}(x) = \beta \left( \frac{x}{\sqrt{(t^2 - x^2)}} \right) \text{ for } |x| < t.$$ 

**Corollary 1'.** Let $\gamma(x, t), B$ be as above and let $u(x, t), \phi(y), \psi(y)$ be as in Theorem 1. Then

$$\lim_{t \to \infty} \left\| \gamma u(\cdot, t) \right\|_2^2 = \int_B |\psi(-\lambda)|^2 + |\phi(\lambda)|^2 \, d^n\lambda.$$ 

**Remark.** Similar expressions can be obtained for the kinetic and potential energies $\left\| \gamma u_1 \right\|_2^2, \left\| \gamma u_2 \right\|_2^2 + \sum \left\| \gamma u_{x_1} \right\|_2^2$ since $u_t$ and $u_{xt}$ can also be written in the form of equation (1). This gives an extension of the Virial theorem stated in Brodsky [1].

**Proof.** By Theorem 1 it suffices to prove the corollary with $u(x, t)$ replaced by $a(x, t)$. When $|x| < t$ we have

$$\gamma(x, t)a(x, t) = \beta \left( \frac{x}{\sqrt{(t^2 - x^2)}} \right) \left\{ e^{i\theta(x, t)} \phi \left( \frac{-x}{\sqrt{(t^2 - x^2)}} \right) + e^{-i\theta(x, t)} \phi \left( \frac{x}{\sqrt{(t^2 - x^2)}} \right) \right\} \rho(x, t),$$

and hence, by (6),

$$\left\| \gamma a(\cdot, t) \right\|_{2, t}^2 = \int \beta(\lambda) \left| e^{i\alpha(\lambda, t)} \psi(-\lambda) + e^{-i\alpha(\lambda, t)} \phi(\lambda) \right|^2 \, d^n\lambda.$$ 

Since $|e^{i\alpha}\psi + e^{-i\alpha}\phi|^2 = |\psi|^2 + |\phi|^2 + e^{2i\alpha}\psi\phi + e^{-2i\alpha}\phi\psi$ the corollary follows from

**Lemma 2.** For $f \in L^1(\mathbb{R}^n)$ define $I_f(t) = \int \exp \left[ it/\sqrt{(1 + \lambda^2)} \right] f(\lambda) \, d^n\lambda$. Then $\lim_{|t| \to \infty} I_f(t) = 0$.

**Proof.** Since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and since $|I_f(t) - I_g(t)| \leq \|f-g\|_1$ it suffices to prove the lemma when $f$ is continuous and has compact support. Switching to spherical coordinates gives

$$I_f(t) = \int_0^\infty \exp \left[ it/\sqrt{(1 + r^2)} \right] F(r) \, dr, \quad F(r) = r^{n-1} \int_{|\nu|=1} f(r\nu) \, dS(\nu).$$

Applying the change of variable $u = 1/\sqrt{(1+r^2)}$ yields
\[ I_f(t) = \int_0^1 e^{iu\phi(u)} du, \quad \phi(u) = F\left(\frac{\sqrt{(1 - u^2)}}{u}\right) \frac{1}{u^2\sqrt{(1 - u^2)}}. \]

Since \( \sqrt{(1 - u^2)} \phi(u) \in C_c((0, 1]) \), the lemma follows from the Riemann-Lebesgue theorem.

**Remark on \( L^p \) behavior.** By Hölder's inequality

\[
\| \gamma u(\cdot, t) \|_2 \leq \| \gamma(\cdot, t) \|_q \| \gamma u^2(\cdot, t) \|_{2q'} = (\| \gamma(\cdot, t) \|_1)^{1/q'} \| \gamma u(\cdot, t) \|_{2q'}^2
\]

where \( 1 - 1/q = 1/q' = 2/p \). Thus Corollary 1' gives a lower bound for \( \lim \inf_{t \to 0} t^{-n/2-n/p} \| \gamma u(\cdot, t) \|_p \), \( p \geq 2 \), which is positive unless \( \phi(K) \) and \( \psi(-\lambda) \) vanish for \( \lambda \in B \).

On the other hand, by using Corollary 2(i) and the estimate

\[
\| u(\cdot, t) \|_p = O(t^{-n/2+n/p}), \quad \| u(\cdot, t) - a(\cdot, t) \|_p = o(t^{-n/2+n/p})
\]

as \( t \to 0 \), provided \( \phi \) and \( \psi \) satisfy the condition of Corollary 2.

**References**


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